Capital Allocation for Property-Liability Insurers: A Catastrophe Reinsurance Application

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Introduction

In their important paper, *Line-by-Line Surplus Requirements for Insurance Companies*, Stuart Myers and James Read[1] have developed an economically sound method for allocating capital to lines of insurance. The purpose of my paper is to extend their model and apply it to the pricing of catastrophe insurance by layer of coverage. I also show the connection between capital allocation and risk loads by layer. With this paper, I hope to introduce the Myers-Read (MR) method and its underlying concepts to a wider, less finance-oriented audience.

The first section of this paper outlines the MR method, showing the relationship of this model to the expected policyholder deficit concept. Also, I show how the underlying MR premise, the constant default ratio, follows from a more fundamental price homogeneity assumption. I further simplify the MR model to incorporate the concept of the *loss beta*. This is a natural extension of the model, since the relevant risk measure for capital allocation depends on the covariance of the losses of a line of business with the total losses for all lines. Another refinement to the MR model is an adjustment to the capital allocation formula to reflect empirical measurement of the loss volatilities and covariances.

The second section of the paper extends the MR method to allocating capital to coverage layers of an individual policy. Given the capital allocated to the entire contract, the further allocation to layer depends on the *layer beta*, a concept analogous to the loss beta. The layer beta is a simple function of the first two partial moments of the loss size distribution. Using the notion of an infinitesimally narrow layer, I show that the layer beta, and therefore the allocated capital, is a monotonic increasing function of the coverage limit.

[1] This paper was developed under the aegis of the Automobile Insurance Bureau of Massachusetts. It has not yet been published.

The third section borrows the powerful risk-neutral probability technique from finance, and applies it to develop risk loads by layer. Having the risk loads by layer is essential to capital allocation by layer, since the capital allocation depends on the market values of losses by layer, not just the expected losses. Particular emphasis is given to calculating risk loads for layers under the lognormal distribution.

The fourth and fifth sections of the paper apply the extended MR model to determine the fair price for a high-layer catastrophe reinsurance treaty. The data used are modified from an actual pricing exercise at my company. An important part of this section is estimating parameters from empirical data.

The sixth and final section discusses key results and suggests areas for future research.

To help the reader follow the mathematics in this paper, I have appended a glossary of notation after Section 6.

Section 1: The Myers-Read Model and Some Extensions

The Role of Capital in Insurance

Capital is defined as the net of total assets over obligations (liabilities) to non-owners. Thus, depending on the accounting measure used, capital could also be called surplus, equity or the market value of the firm. The principal role of capital in insurance is to reduce the impact of insolvency on insured policyholders.

Since policyholders have a preference for protection against default on their claims, they will want insurers to have more capital than less. However, due to the double-taxation of corporate and personal income in the U.S., capital is costly.[2] Therefore, there is a tradeoff

[2] When an insurance company owner contributes a dollar of capital to secure losses against default, the investment return from that dollar is taxed at the corporate level. The net return to the owner is $r_A(1-t)$, where t is the tax rate and r_A is the investment return. However, the owner could get r_A simply by directly investing the dollar in the financial markets. Thus, the owner's opportunity loss is $r_A t$ per dollar invested.

between the cost and benefit of having capital. This tradeoff, in the interplay between insurers, consumers and regulators, determines the level of capital carried by the average insurer.

One-Period Default Model

The cost of an insurer's insolvency is readily determined in a competitive market. Denote the value of the insurer's assets at the beginning of the period by A and the value of the losses (also called liabilities) by L. The amount of capital C is $A-L$. The respective values at the end of the period are A' and L'. At the end of the period, if $L' > A'$, then the insurer is insolvent and the policyholders are short a default amount $L' - A'$. If $L' \leq A'$, then the default amount is zero. Let D represent the market value[3] of the expected default amount and $d = D/L$ the default market value per unit of expected loss.

The expected default is also called the *expected policyholder deficit* in the actuarial literature.[4] Several authors have shown that the market value of the expected amount of default is equivalent to the price of a put option on the insurer's assets. These authors include Butsic (1994), Cummins (1988), Derrig (1989) and Doherty and Garven (1986). The option to default is implicitly given by the policyholders to the owners of the insurance company.

Competitive Market Model

We assume that the market for insurance is competitive: all insurance contracts having identical coverage and identical levels of service will have the same price (called the *fair* price or *fair* premium). Also, the default values for all insurers have a market price. To

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In order to induce the owner to contribute capital, the policyholders must pay an additional amount in their premiums to make up the income tax cost. Appendix 6 has a more complete discussion of income tax costs.

[3] In financial economics, the market value equals the present value, at a risk-free interest rate, of the expected default amount at the end of the period. In this calculation, the expectation is taken with respect to a risk-neutral probability distribution. For more information, see Brealey and Myers (1996) and Panjer (1998).

[4] See Butsic (1994). This term was chosen to distinguish between the cost of insolvency to the policyholder (whose premiums should reflect this cost) and the cost to capital providers, whose liability in a bankruptcy is usually limited.

simplify the analysis, we assume that the only service or quality differentiation between insurers is the amount of default and that premiums are net of expenses. Also assume that prices are homogeneous, so that the market price per unit of insured loss is constant. All insurers have identical capital amounts and line composition. All assets are invested proportionately in the same portfolio of investments, so that all insurers' investment returns will be identical.

For a single insurer, the fair total premium for all policies is $P = L - D + TC$, where T is the present value of income taxes[5] per unit of capital. In words, the market price equals the market value of the loss minus the market value of the default plus the cost of income taxes. The respective loss value, capital and default values for policy *i* are L_i, C_i and D_i. The sum of each of these quantities over all policies equals the respective unsubscripted amount (e.g., $\sum L_i = L$). Denote the capital per unit of loss for the insurer as $c = C / L$, and for the individual policy as $c_i = C_i / L_i$. The fair premium for policy *i* is

$$(1.1) \quad P_i = L_i - D_i + TC_i = L_i - h_i(L_i / L)D + Tc_iL_i$$

where h_i is a constant that determines the share of the total default value belonging to policy *i* and D_i is the portion of total default *D* assigned to policy *i*. If we define $d_i = D_i / L_i$, then $h_i = d_i / d$. We want to find equilibrium values of h_i and c_i.

Notice that, even if each insurer has a different capital and line mix, equation (1.1) still holds, because the market price excluding the cost of default must be unique. The default-free premium component $L_i + Tc_iL_i = P_i + D_i$ will be the same for all insurers selling line *i* (although D_i would depend on company-specific parameters). Regardless of the amount of capital held by the selling insurer, no policyholder would pay a higher price

[5] Appendix 6 derives the result that, for a one-period model, T will equal $tr/[(1+r)(1-t)]$, where t is the income tax rate and r is the one-period default-free interest rate. For more details see Myers and Cohn (1987) and Derrig (1994).

for the same default-free transfer of risk. Thus, the capital allocation factor c_i must be generic to all companies and would represent some sort of industry average capital allocation.

In a competitive market, the price per unit of expected loss will stay constant at the margin, so we have

$$(1.2)\quad \frac{\partial P_i}{\partial L_i}=\frac{P_i}{L_i}, or \quad 1-\frac{\partial}{\partial L_i}[h_i\frac{L_i}{L}D]+Tc_i \quad = \quad 1-h_i\frac{D}{L}+Tc_i.$$

This gives $\frac{\partial}{\partial L_i}\left(\frac{L_i}{L}D\right)=\frac{D}{L}$. Since $\partial L/\partial L_i=1$, we get $\frac{\partial}{\partial L_i}\left(\frac{L_i}{L}D\right)=$ $\frac{L[L_i(\partial D/\partial L_i)+D]-DL_i}{L^2} = \frac{D}{L}$, which reduces to

$$(1.3)\quad \frac{\partial D}{\partial L_i}=\frac{D}{L}=d.$$

Thus, the overall default ratio d remains constant with a marginal shift in the mix of policies. Notice that we do not have to assume[6] that d remains constant, since it follows directly from the competitive market assumptions that premiums properly reflect the market value of default and that the price per unit of coverage is constant.

We see here that the default allocation between the lines is irrelevant to determining the capital allocation in a competitive market. The default allocation parameters h_i do not appear in equation (1.3), which governs the capital allocation, although they appear in the premium. Also, the income tax cost factor T does not enter into the capital allocation.

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[6] Myers and Read start from this assumption, as I did in an earlier paper [Butsic (1994)].

This result establishes an economic rationale for keeping the default ratio d constant in regulatory and pricing applications.

Deriving a Capital Allocation Formula

The expected value of default depends on the specific probability distributions governing the change in value of the insurer's assets and liabilities. A natural candidate is the lognormal distribution, since it is commonly used in asset valuation (e. g., the Black-Scholes option-pricing model).

The lognormal density function is

$$(1.4) \quad f(z) = \frac{1}{\sqrt{2\pi}\, vz} \exp\left(\frac{[\ln(z) - \mu]^2}{2v^2}\right) dz .$$

Here, $x \geq 0$; μ is called the location parameter and v is called the dispersion parameter, or *volatility*. The mean of the lognormal distribution is $Y = \exp(\mu + v^2/2)$ and the variance is $Y^2[\exp(v^2) - 1]$. Thus the coefficient of variation (CV), or standard deviation divided by the mean, is simply related to the volatility by $CV^2 = [\exp(v^2) - 1]$.

As shown by Myers and Read, if the assets and liabilities have respective volatilities v_A and v_L, along with a covariance parameter v_{AL}, then the value of the default depends on a single lognormal volatility v, such that

$$(1.5) \quad v^2 = v_A^2 - 2v_{AL} + v_L^2 .$$

The sign of the covariance parameter is negative because a positive correlation (e. g., a simultaneous increase/decrease in the value of assets and liabilities) will reduce the total volatility v. The default value[7] is found directly by the Black-Scholes option-pricing formula:

(1.6) $D = LN(y+v) - AN(y) = L[N_1 - (1+c)N_2]$.

Here, $y = -\ln(1+c)/v - v/2$ and $N(y)$ denotes the cumulative standard normal probability distribution. Also, $N_1 = N(y+v)$ and $N_2 = N(y)$.

To find the capital allocation for each policy, we use equation (1.3). Taking the partial derivative of D with respect to L_i in equation (1.6) and setting the result equal to equation (1.3) yields

(1.7) $$\frac{\partial D}{\partial L_i} = L\left(\frac{\partial N_1}{\partial L_i} - (1+c)\frac{\partial N_2}{\partial L_i} - N_2\frac{\partial(1+c)}{\partial L_i}\right) + [N_1 - N_2(1+c)]\frac{\partial L}{\partial L_i} = N_1 - N_2(1+c).$$

Note that $L = \sum L_i$ and $c = \sum(c_i L_i / L)$. Thus, $\partial L/\partial L_i = 1$ and $\partial c/\partial L_i = (c_i - c)/L$, giving

(1.8) $$(c_i - c)\frac{N(y)}{L} = \frac{\partial}{\partial L_i}[N(y+v)] - (1+c)\frac{\partial}{\partial L_i}[N(y)].$$

We also have $\partial[N(u)]/\partial L_i = n(u)[\partial u/\partial L_i]$ for a variable u and $n(y+v) = (1+c)n(y)$, where $n(y)$ is the standard normal density. After substituting these expressions into equation (1.8), we get

(1.9) $$c_i = c + \frac{(1+c)Ln(y)}{N(y)}\frac{\partial v}{\partial L_i}.$$

[7] Here, we are evaluating an exchange option (i.e., to trade one stock for another) whose worth is analogous to that of the difference between insurer's asset value (the first stock) and its liability value (the second stock). Panjer (1998) p. 481-484, shows that the exchange option value can be found directly from the Black-Scholes model using the composite volatility v.

Volatility, Line Mix and Loss Beta

To derive a practically useful result from equation (1.9), we need to determine $\partial v / \partial L_i$, or how the overall company volatility v changes with a change in L_i.

We have assumed that the sum of individual loss values is lognormal. However, the individual loss values will not be lognormal, since the sum of lognormal variables is not lognormal (the product is, however). For the total of all lines, denote the *CV* of the loss by σ_L. Thus $\sigma_L^2 = \exp(v_L^2) - 1$. If the volatility is small, then $v_L \cong \sigma_L$.

We assume that liabilities (losses) have distinct CV values σ_i for each policy *i*, and that the losses for each pair of policies may be correlated. The total loss CV σ_L is determined by

(1.10) $\sigma_L^2 = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij} = \exp(v_L^2) - 1,$

where $w_i = L_i / L$ is the weight of the losses from the *i*th policy and ρ_{ij} is the correlation coefficient between losses of policy *i* with those of policy *j*.

The volatility of the assets, v_A, does not depend on the mix of lines or policies, since the investment portfolio composition does not change with the volume of assets. However, the covariance parameter v_{AL} will vary with the mix. To simplify the analysis, we normalize the asset and liability values: let $\tilde{a}$ be the asset value divided by its mean and $\tilde{\ell}$ the liability value divided by its mean. In other words, $\tilde{\ell} = \tilde{L} / L$. (The tilde denotes a random variable and the variable with no tilde represents its expected value.) Then the expected value of $\tilde{a}$ is $E[\tilde{a}] = \exp[\mu_a + v_A^2 / 2] = 1$. Here, $\mu_a = -v_A^2 / 2$ is the lognormal location parameter for the normalized asset value. Similarly, we have $\mu_\ell = -v_L^2 / 2$.

The covariance between $\tilde{a}$ and $\tilde{\ell}$ is $\sigma_{AL} = E[\tilde{a}\,\tilde{\ell}] - E[\tilde{a}]\,E[\tilde{\ell}] = \exp[E[\tilde{a}\,\tilde{\ell}] - 1$. Since the product of the two lognormal variables is another lognormal variable with a location parameter equal to the sum of the individual location parameters and volatility equal to the standard deviation of the underlying joint normal distribution, we have[8] $v^2 = v_A^2 + 2v_{AL} + v_L^2$. Therefore, $E[\tilde{a}\,\tilde{\ell}] = \exp[\mu_a + \mu_\ell + v^2/2] = \exp[-v_A^2/2 - v_L^2/2 + v^2/2] = \exp[v_{AL}]$ and we get

(1.11) $\sigma_{AL} = \exp[v_{AL}] - 1.$

Breaking $\tilde{\ell}$ into its components, we have $\tilde{\ell} = \sum_i w_i\,\tilde{\ell}_i$, where $\tilde{\ell}_i = \tilde{L}_i / L_i$. Let ω_i be the correlation coefficient between $\tilde{\ell}_i$ and $\tilde{a}$. Then

(1.12) $\sigma_{AL} = Cov(\tilde{a}, \tilde{\ell}) = \sum_i w_i\,Cov(\tilde{a}, \tilde{\ell}_i) = \sigma_A \sum_i w_i \omega_i \sigma_i.$

Finally, we can determine $\partial v / \partial L_i$ in terms of the CV values and correlation coefficients of the individual policies or lines of business. Taking the partial derivative of equation (1.5) with respect to L_i, we get

(1.13) $$\frac{\partial v}{\partial L_i} = \frac{v_L}{v}\frac{\partial v_L}{\partial L_i} - \frac{v_{AL}}{v}\frac{\partial v_{AL}}{\partial L_i}.$$

Noting that $\partial w_i / \partial L_i = (1 - w_i)/L$ and $\partial w_j / \partial L_i = -w_j / L$ for $i \neq j$, we take the derivative of equation (1.10) with respect to L_i getting

[8] Notice that the sign of the covariance term here is positive. In equation (1.5), it is negative, reflecting the fact that we are evaluating an option whose value is the *difference* between an insurer's assets and liabilities.

(1.14) $$\frac{\partial v_L}{\partial L_i}=\frac{\sigma_{iL}-\sigma_L^2}{v_L L(1+\sigma_L^2)},$$

where $\sigma_{iL}=\sigma_i\sum_k w_k\sigma_k\rho_{ik}$ is the covariance of the *i*th policy losses with the entire portfolio of losses. Now we equate the derivatives of equations (1.11) and (1.12) with respect to L_i, getting

(1.15) $$\frac{\partial v_{AL}}{\partial L_i}=\frac{\omega_i\sigma_A\sigma_i-\sigma_{AL}}{v_{AL}L(1+\sigma_{AL})}=\frac{\sigma_{iA}-\sigma_{AL}}{v_{AL}L(1+\sigma_{AL})}.$$

Here, σ_{iA} is the covariance of the loss CV of policy *i* with the asset CV. Substituting equations (1.14) and (1.15) into equation (1.13) and that result into equation (1.9) we get

(1.16a) $$c_i=c+\frac{(1+c)n(y)}{N(y)v}\left[\frac{\sigma_{iL}-\sigma_L^2}{(1+\sigma_L^2)}-\frac{\sigma_{iA}-\sigma_{AL}}{(1+\sigma_{AL})}\right].$$

Equation (1.16a) is equivalent to the Myers-Read result. The actual MR formula is, in my notation,

(1.16b) $$c_i=c+\frac{(1+c)n(y)}{N(y)v}\left[(v_{iL}-v_L^2)-(v_{iA}-v_{AL})\right].$$

The advantage of equation (1.16a) over equation (1.16b) is that σ_{iL} and σ_{iA} are directly measurable from empirical data, while their volatility counterparts are not. For practical applications, the above MR result can be simplified, as I show next.

Simplifying the MR Model

Define the *loss beta* for the policy as $\beta_i=\sigma_{iL}/\sigma_L^2$. Since $\sigma_{iL}=\rho_{iL}\sigma_i\sigma_L$, where ρ_{iL} is the correlation between policy or line *i* losses and all losses, we also have

$\beta_i = \rho_{iL}(\sigma_i / \sigma_L)$. This is a parallel definition to the beta used in the Capital Asset Pricing Model (CAPM). Notice that if losses for policy (or line) i are independent of the other policies, then $\beta_i = w_i(\sigma_i^2 / \sigma_L^2)$. Similarly, for the policy loss-to-asset relationship, we define $\gamma_i = \sigma_{iA} / \sigma_{AL}$. Thus, equation (1.16a) becomes

$$(1.17)\quad c_i = c + \frac{(1+c)n(y)}{N(y)v}\left[\frac{(\beta_i - 1)\sigma_L^2}{(1+\sigma_L^2)} - \frac{(\gamma_i - 1)\sigma_{AL}}{(1+\sigma_{AL})}\right].$$

Notice that, since $\sum w_i \beta_i = 1$ and $\sum w_i \gamma_i = 1$, equation (1.17) gives $\sum w_i c_i = c$, which correctly reproduces the relationship between c_i and c. Also, if $\beta_i = 1$ and $\gamma_i = 1$, we get $c_i = c$. It is important to notice that, under this model, the company's total capital C is allocated to each policy in an amount C_i, with no overlap or shortfall. All the individual capital amounts add to the total capital.

In general, the covariance between asset and liability values (σ_{AL}) will be small relative to σ_L^2, so the covariance can be ignored[9] without introducing much error. From this point forward, we assume that σ_{AL} is zero. Given this assumption, equation (1.17) reduces to

$$(1.18)\quad c_i = c + (\beta_i - 1)Z,$$

with $Z = \frac{(1+c)n(y)}{N(y)v}\frac{\sigma_L^2}{(1+\sigma_L^2)} \cong (1+c)\frac{n(y)}{N(y)}\frac{\sigma_L^2}{\sigma}$. Thus, for a given level of capital and a fixed total loss and asset variability, equation (1.18) shows that the capital allocation to a policy or line is a linear function of its loss beta. The slope Z acts as a leveraging factor: the higher/lower the value of Z, the farther away /closer to the average capital ratio the

[9] In Massachusetts rate filings, the underwriting beta is assumed to be about –0.20. The average property-liability insurer has only about 20% of its investments in stocks, and the other assets (mostly bonds) have a negligible correlation with stock market returns. Thus, σ_{AL} is about –0.04 times the squared annual stock CV σ_S^2 (about 0.030), which is roughly equal to σ_L^2 (about 0.010 to 0.040).

line's capital ratio will be. Because of its importance in capital allocation, we name it the *capital allocation factor.*

In a ratemaking application where we are trying to estimate a fair price for policy *i*, the factor *Z* will be a constant that depends on the equilibrium capital ratio *c* and all-lines loss and asset volatilities. These will be generic parameters based on industry data.

Since $\beta_i = \rho_{iL}(\sigma_i / \sigma_L)$, a line with a high loss CV relative to that of all lines can still require less capital than the average line if its correlation with the total losses is small. This is a consequence of diversification that explains why multi-line companies write property catastrophe insurance.

Another interesting consequence occurs when a line has a low loss CV and also a low correlation with other lines' losses. In this case, it is possible for the allocated capital to be negative.[10]

Capital Allocation and Loss Beta Within a Line of Business

If we know the capital ratio for a particular line of business, along with the covariances of losses within the line, then we can produce a formula for capital allocation within the line. Appendix 1 develops this formula, shown as equation (1.19).

$$(1.19) \quad c_i = c_k + \left[\frac{\rho_i}{\rho_{ik}\rho_k} \beta_{ik} - 1 \right] \beta_k Z .$$

Here, c_k is the capital ratio for the line of business containing policy *i*, β_k is the loss beta for the line relative to all lines, β_{ik} is the loss beta for the policy relative to the line containing it, ρ_i is the correlation between the policy losses and losses of all lines (including line *k*), ρ_{ik} is the correlation between the policy losses and losses of the line *k*,

[10] This does not present a problem using the financial pricing model of equation (1.1). However, as discussed in Section 6, it means that a return on equity pricing model cannot work.

and ρ_k is the correlation between the line losses and losses of all lines (including line k). If the covariance of losses in line k with all *other* losses (excluding line k) is zero, Appendix 1 shows that equation (1.19) simplifies to

$$(1.20) \quad c_i = c_k + [\beta_{ik} - 1]\beta_k Z.$$

This simpler formula will apply in the case of most property lines, whose losses are presumed to be uncorrelated with other non-property lines.

Capital Allocation with Alternative Probability Distributions

The Myers-Read method is not dependent on the lognormal distribution. For example, assume that the loss and asset values are normally distributed with the same CV as in the lognormal case and $\sigma_{AL} = 0$. With a derivation similar to that of the lognormal case, we get

$$(1.21) \quad c_i = c + (\beta_i - 1)\frac{\sigma n(c/\sigma)\sigma_L^2}{\sigma N(-c/\sigma) - (1+c)\sigma_A^2}.$$

Notice that this result also shows a linear relationship with the loss beta.

Numerical Example of MR Method

An insurer has three lines whose loss values in total are lognormally distributed. The respective loss values for the three lines are 500, 400 and 100, for a total of 1000. The respective CV values are 0.2, 0.3 and 0.5. The correlation between line 1 and line 2 is $\rho_{12} = 0.75$, with the other interline correlations being zero. Thus, the total loss CV is 0.2119 and the loss volatility is 0.2096.

The insurer's capital is 500, giving $c = 0.500$. Asset values are also lognormal with annual asset CV of 0.0700 and volatility of 0.0699. Thus, the total volatility is

$v = 0.2209$ and $Z = 0.6784$. The loss betas and the capital allocation to line are shown in Table 1.1.

Table 1.1

Loss Beta and Capital Allocation for Numerical Example

	Liability Value	Loss CV	Loss Beta	Capital/ Liability	Capital
Line 1	500	0.2000	0.8463	0.3957	197.87
Line 2	400	0.3000	1.3029	0.7055	282.19
Line 3	100	0.5000	0.5568	0.1993	19.93
Total	1000	0.2119	1.0000	0.5000	500.00

Notice that even though line 3 has the highest loss variability, it has the lowest beta, and therefore the smallest capital ratio. This occurs because line 3 is uncorrelated with the other two lines and its volume is small compared to the total.

Section 2: Capital Allocation to Coverage Layers

There is no difference conceptually between separate policies and separate coverage layers within the same policy. Capital can be allocated to layers in the same manner as it is to individual policies or lines. We will apply equation (1.20), treating the individual policy as a "line" and the layer as a "policy."

Layer Beta

Let $\tilde{X}$ be the loss (having expected value *X*) for an individual policy and let $\tilde{X}(a,b)$ be the amount of loss covered in the layer with lower limit *a* and upper limit *b*. When there is no upper limit, the covered loss is denoted by $\tilde{X}(a,\infty)$. If $\tilde{X} \le a$, then $\tilde{X}(a,b) = 0$. If $a \le \tilde{X} \le b$, then $\tilde{X}(a,b) = \tilde{X} - a$. If $b \le \tilde{X}$, then $\tilde{X}(a,b) = b - a$. It is easy to show that the entire loss equals the sum of the losses covered by the three layers defined by *a* and *b*:

$$(2.1) \quad \tilde{X} = \tilde{X}(0,a) + \tilde{X}(a,b) + \tilde{X}(b,\infty).$$

It is useful here to define the partial *n*th moment (evaluated at *u*) of $\tilde{X}$, as

$$(2.2) \quad E_n(u) = \int_u^\infty x^n f(x)\,dx.$$

Notice that $E_0(u) = 1 - F(u)$, where $F(u)$ is the cumulative distribution function. The expected value of $\tilde{X}(a,b)$ is

$$(2.3) \quad X(a,b) = E_1(a) - E_1(b) - [a\,E_0(a) - b\,E_0(b)].$$

As shown in Appendix 2, the covariance between $\tilde{X}(a,b)$ and $\tilde{X}$ is

$$(2.4) \quad Cov[\tilde{X}(a,b), \tilde{X}] = E_2(a) - E_2(b) - [a\,E_1(a) - b\,E_1(b)] - X(a,b)\,X.$$

Thus the loss beta of the layer, relative to the entire policy, is

$$(2.5)\quad \beta(a,b) = \frac{Cov[\{\tilde{X}(a,b)/X(a,b)\},(\tilde{X}/X)]}{Var(\tilde{X}/X)} = \frac{1}{w(a,b)}\frac{Cov[\tilde{X}(a,b),\tilde{X}]}{Var(\tilde{X})},$$

Where $w(a,b) = X(a,b)/X$ is the weight of the layer relative to the entire loss. Notice that the loss beta is calculated using the CV measure, in which the standard deviation of the loss is divided by the mean.

Assume the case where the covariance of losses in line k (the line whose expected loss is X) with all other lines is zero. Applying equation (1.20) we get a formula for the capital ratio of the layer from a to b.

$$(2.6)\quad c(a,b) = c_k + [\beta(a,b)-1]\beta_k Z \;=\; c_k + [\beta(a,b)-1]\,Z_k,$$

where $Z_k = \beta_k Z$.

For the lognormal distribution with location parameter μ and volatility υ,

$$(2.7)\quad E_n(u) = E[\tilde{X}^n]N\left(\frac{\mu - \ln(u)}{\upsilon} + n\upsilon\right).$$

Equation (2.7) can be used in equations (2.3) and (2.4) to produce the layer beta $\beta(a,b)$ for the lognormal distribution.

Numerical Example

A policy's losses are lognormal with mean 100 and standard deviation of 50. Thus, the second loss moment is 12500, with $\mu = 4.4936$ and $\upsilon = 0.4724$. Consider the layer whose lower coverage limit is 100 and upper limit is 200. From equation (2.7),

$E_0(100) = 0.4066$, $E_0(200) = 0.0442$, $E_1(100) = 59.34$, $E_1(200) = 10.91$, $E_2(100) = 9508.81$ and $E_2(200) = 2799.91$.

Thus, $X(a,b) = 16.61$, $w(a,b) = 0.1661$, $Cov[\tilde{X}(a,b), \tilde{X}] = 1297.46$ and $\beta(100{,}200) = 3.125$. Table 2.1 below summarizes the results for all three layers.

Table 2.1

Example Results by Layer

Layer	*0 to a*	*a to b*	*b to Infinity*	*Total*
Expected Covered Loss	81.32	16.61	2.07	100.00
Weight	0.8132	0.1661	0.0207	1.0000
Covariance	791.92	1297.46	410.62	2500.00
Beta	0.389	3.125	7.948	1.000

Notice that the weighted average of the betas is 1, and that the betas increase with the level of the layer.

General Layer Beta Properties

We can gather further insight into capital requirements by coverage layer by examining the behavior of infinitesimally narrow layers. Consider the layer from x to $x+\Delta$, where Δ is so small that no loss value can occur between x and $x+\Delta$. Then the coverage is simply Δ for any loss *greater than x.* Thus, the *right tail* of the probability distribution, or $G(u) = 1 - F(u)$ for a cumulative probability function $F(u)$, is important in measuring risk associated with layer coverage.

As discussed in Section 3, using the right-tail probability is a simple way to generate risk loads by layer.

Define the *point beta* of the loss amount x as $\beta(x) = \lim_{\Delta \to 0} \beta(x, x+\Delta)$. Appendix 3 develops the general formula, for any well-defined loss distribution with $x \geq 0$:

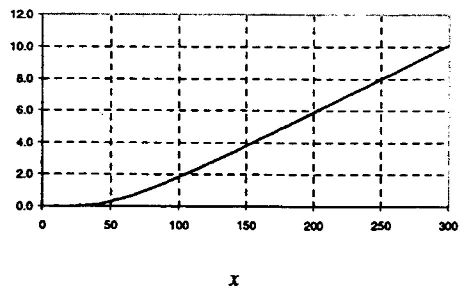
(2.8) $\beta(x) = \frac{1}{s^2}\left(\frac{E_1(x)}{X\,E_0(x)} - 1\right).$

Here, s is the CV of the loss, or $s^2 = Var(\tilde{X})/X^2$. Notice that $\beta(0) = 0$, since $E_1(0) = X$ and $E_0(0) = 1$. Also, as shown in Appendix 3, the slope of $\beta(x)$ is positive, so the layer beta, and thus the layer capital, is a strictly increasing function of the level of the layer.

In the above numerical example, we have $\beta(100) = 1.837$ and $\beta(200) = 5.869$. The layer beta of 3.125 is between these two values. In this example, Figure 2.1 shows that, above the mean, the point beta is approximately linear with the loss amount.

Figure 2.1

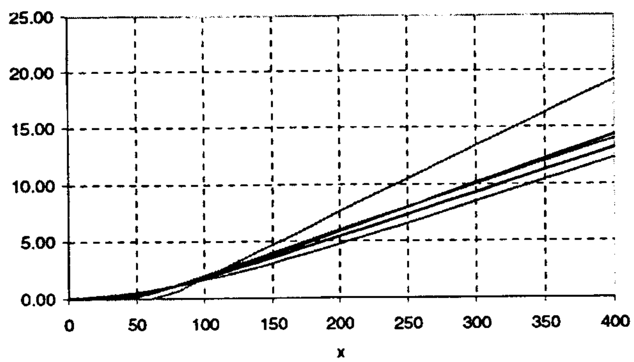
Approximate Linearity of Point Beta for Numerical Example



Appendix 3 develops formulas for the point betas for several loss distributions: normal, lognormal, gamma, Pareto and exponential. Figure 2.2 below displays point betas for the numerical example. For this calculation, all distributions have a mean of 100 and a standard deviation of 50 (CV of 0.50).

Figure 2.2

Point Betas for Numerical Example Using Various Loss Distributions



Legend: right hand side (x = 400), top to bottom

Pareto
Lognormal
Exponential
Gamma
Normal

For all these distributions, the point beta is approximately linear above the mean. In fact, the Pareto and exponential distributions[11] give an exact linear relationship.

[11] Note that the log of an exponential random variable has the Pareto distribution. Also, as shown in Appendix 4, the versions of these two distributions are defined to allow us to vary the mean and standard deviation. Thus, they are not defined at zero.

Section 3: Risk Loads by Layer Using Risk-Neutral Probability Distributions

In an economically valid model, capital allocation depends on the *market values* of losses, not just the expected losses. Section 2 has shown how capital can be allocated to a layer based on the moments and covariances of the pure losses. However, this development is incomplete.

Calculating market values by layer is equivalent to determining risk loads by layer, as shown below. The application of risk loads by layer is at least as important as capital allocation by layer. However, both concepts are addressed by modern financial theory. Appendix 4 contains a more detailed development of the financial theory underlying risk loads.

Risk Loads and Risk-Neutral Probability Measures

In section 2, X is the expected value of gross losses subject to a particular coverage, before applying the layer limits. Assuming that X represents the pure (non-risk-adjusted) loss values at the end of one period, the market value of the gross losses at the beginning of the period can be expressed as $X(1+\lambda)/(1+r)$, where λ is the *risk load* and r is the one-period default-free (riskless) interest rate. In other words, $X(1+\lambda)$ is the market, or fair value of the liability viewed at the end of the period. Since policyholders, along with investors, are assumed to be risk averse, λ will usually be positive.

In finance, the market valuation at the end of a period can be expressed using the powerful *risk-neutral probability* concept.[12] Here, each actual probability $f(x)$ of a specific loss amount x is replaced by an altered probability (the risk-neutral probability) $\hat{f}(x)$. The risk-neutral probability value is chosen so that, at the end of the period, the investor is indifferent between having a certain amount $\hat{X}$ (the certainty equivalent value) or the actual random amount $\tilde{X}$. The certainty equivalent amount is the above market value loaded for risk:

[12] See Brealey and Myers (1996) or Panjer (1998) for a good basic explanation of this concept.

$$(3.1)\quad \hat{X} = X(1+\lambda) = \int_0^\infty x\hat{f}(x)\,dx.$$

Denote by $\hat{X}(a,b)$ the certainty equivalent loss covered by the layer from a to b. Extending the risk load definition, we have $1+\lambda(a,b) = \hat{X}(a,b)/X(a,b)$ and

$$(3.2)\quad \hat{X}(a,b) = \int_a^b (x-a)\hat{f}(x)\,dx + (b-a)\int_b^\infty \hat{f}(x)\,dx.$$

Notice that, if we separate the entire range of loss into layers, the transformed probability density guarantees that the certainty equivalent loss amounts by layer add to the total certainty equivalent loss. This in turn insures that the risk loads by layer add to the total risk load.[13] Venter (1991) uses the no-arbitrage principle to show that the only risk loads that satisfy layer additivity must result from a transformed probability distribution.

Define the *point risk load* as $\lambda(x) = \lim_{\Delta\to 0}\lambda(x, x+\Delta)$. This is similar to the point beta concept from section 2. Define the *right-hand tail* of a cumulative probability function $F(u)$ as $G(u) = 1 - F(u)$. $\hat{G}(u)$ is the right-hand tail of the corresponding risk-neutral distribution. Appendix 4 shows that, for any probability distribution, the point risk load (abbreviated as PRL) is

$$(3.3)\quad \lambda(x) = \frac{\hat{G}(x)}{G(x)} - 1.$$

This is a compact and powerful result. The PRL at loss size x can be simply expressed as the ratio of the right-hand tails of the cumulative risk-neutral and actual loss distributions, minus 1.

---

[13] Other methods for calculating risk loads do not satisfy the additivity principle, which is essential to a market price determination. In particular, variance-based risk loads such as those used by Meyers (1991) and Miccolis (1977) do not have value additivity.

Equation (3.3) has a form similar to that of the point beta in equation (2.8). Since $\hat{G}(0) = G(0) = 1$, we get $\lambda(0) = 0$. This result is intuitively plausible, since a narrow layer at zero of width Δ will almost certainly produce a covered loss of Δ. Thus, there is no risk and there should be no risk load. At the other extreme, as x approaches infinity, many types of risk-neutral density transformations will produce infinitely large point risk loads.

Wang (1998) uses a proportional hazard (PH) transform to determine the risk-neutral probability distribution. In the above notation, the transformed distribution is obtained by $\hat{G}(x) = [G(x)]^q$, with $0 < q \leq 1$. The PH transform can readily be generalized so that, instead of being constant, the exponent q is a function of x:

$$(3.4) \quad \hat{G}(x) = [G(x)]^{q(x)},$$

with $0 \leq q(x) \leq 1$. Appendix 4 shows that any risk-neutral distribution giving a positive point risk load for each x can be generated by a suitable function $q(x)$. See figure 3.1 for the implied generalized PH transform for a lognormal distribution.

Risk Loads for the Lognormal Distribution

A classic application of the risk-neutral probability measure is the Black-Scholes option-pricing model. Instantaneous price changes (relative to the current price) are assumed to drift according to geometric Brownian motion (GBM), which makes the relative changes in stock prices over any period length lognormally distributed.

In the case of loss values behaving like stock values (having GBM), the risk-neutral distribution is simply another lognormal distribution with a shifted location parameter but with the identical volatility as the actual loss distribution.[14] Denoting the shifted location parameter by μ', Appendix 4 shows that

[14] See Panjer (1998), Chapter 10 for a discussion of the parameter shift.

(3.5) $1+\lambda = \exp(\dot{\lambda}) = \exp(\mu' - \mu)$.

Thus, the exponential form of the risk load, $\dot{\lambda}$, equals the difference in the location parameters $\mu' - \mu$.

Varying the Lognormal PRL for Higher Layers

As discussed in Appendix 4, the GBM may not apply for the valuation of property catastrophe losses. Given the same overall risk load for a distribution, risk loads for high layers may be greater in the case of catastrophe losses compared to the case where losses followed GBM. This extra loading appears to be present in the case of actual ceded catastrophe reinsurance market prices seen at my own company.

An easy way to vary the risk load is by using a suitable generalized PH transform $q(x)$. A simple, two-parameter PH transform is the *fractional* transform

(3.6) $q(x) = \dfrac{qx}{x+m}$,

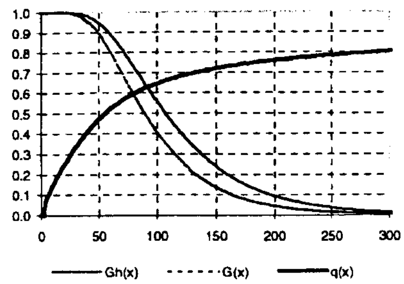
with $m \geq 0$ and $0 \leq q \leq 1$. If m is zero, equation 3.6 equals the constant PH transform. By choosing q and m so that λ remains constant, we can vary the slope of $\lambda(x)$ to a great extent, as shown in the numerical illustration below.

Numerical Example of Lognormal Risk Load by Layer

Let the expected loss be 100 with a CV of 0.50, the same as the example in Section 2. The certainty equivalent expected loss is 120, giving an overall risk load of 0.20. Assume that the risk-neutral probability is determined by a simple location parameter shift (LPS). Figure 3.1 below compares actual and transformed right hand tails $G(x)$ and $\hat{G}(x)$, denoted by Gh(x), as well as the implied generalized PH transform $q(x)$.

Figure 3.1

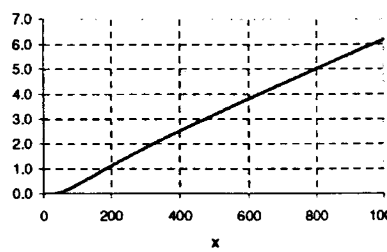
Actual vs. Risk-Neutral Right-Tail Probability Distribution
And Implied Generalized PH Transform
For Numerical Example



Notice that the PH transform lies between 0 and 1. Thus, the resulting point risk load is non-negative at all loss sizes, as shown by Figure 3.2:

Figure 3.2

Point Risk Load Using Location Parameter Shift
For Numerical Example

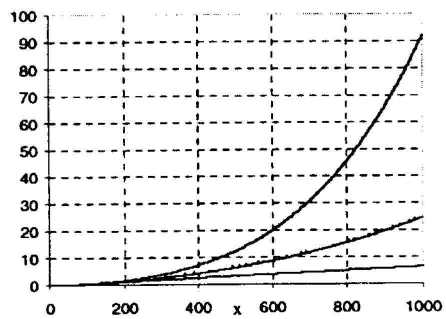


The PRL at the mean is 0.376, or about twice the overall risk load of 0.20. The PRL at loss size 1000 is 6.17, which is over 30 times the overall risk load.

Figure 3.3 below shows the PRL for three values of m using the fractional transform of equation 3.6. All cases have the same overall risk load of 0.20. The three pairs of respective q and m values are (0.7102, 0), (0.8082, 20) and (0.9056, 40).

Figure 3.3

Point Risk Load for Fractional Transform by Varying m Parameter
For Numerical Example



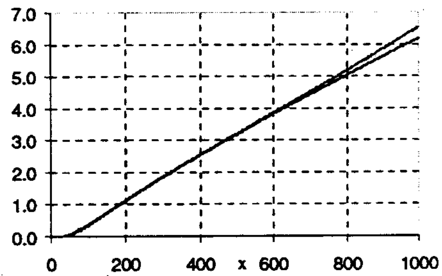
Top line: $m = 0$
Middle line: $m = 20$
Bottom line $m = 40$

For $m = 0$, the PRL at loss size 1000 is 92.14, or about 14.9 times the PRL using the lognormal parameter shift. However, the PRL at lower loss levels is reduced to compensate: the PRL at the mean is 0.298, compared to the 0.376 using the LPS transform.

Figure 3.4 compares the PRL using the fractional transform with $m = 40$ to that using the lognormal LPS.

Figure 3.4

Point Risk Load Using Location Parameter Shift vs. Fractional Transform with m = 40
For Numerical Example



Top line Fractional transform
Middle line: Location parameter shift

Notice here that the fit is quite close. By using $m = 40.7$, the maximum difference between the two PRL functions over the range $0 \leq x \leq 1000$ is about 0.065, occurring when $x = 1000$. Thus, the fractional transform is a handy way to vary the risk load by layer to obtain risk loads equal to or greater than those obtained from shifting the location parameter.

Calculating Betas and Capital by Layer with Risk-Neutral Probabilities

Continuing the above numerical example, we first assume no risk load. Table 3.1 calculates the layer betas for various layers, whose lower limit is a and upper limit is b.

Table 3.1

Calculation of Layer Betas for Numerical Example
$\lambda = 0$

a	b	$E_0(a)$	$E_1(a)$	$E_2(a)$	$X(a,b)$	$Cov[X,X(a,b)]$	$\beta(a, b)$
0	100	1.0000	100.00	12500.0	81.33	791.9	0.39
100	200	0.4066	59.34	9508.8	16.61	1297.5	3.13
200	300	0.0442	10.91	2799.9	1.79	326.1	7.27
300	400	0.0052	1.83	661.6	0.23	65.7	11.56
400	500	0.0008	0.35	162.5	0.04	14.1	15.88
500	Infinity	0.0001	0.08	43.5	0.01	4.7	21.52
0	Infinity				100.00	2500.0	1.00

We next calculate the layer betas using the 0.20 overall risk load and a location parameter shift, shown in table 3.2.

Table 3.2

Calculation of Layer Betas for Numerical Example
$\lambda = 0.20$; *Point Risk Load by Location Parameter Shift*

a	b	$E_0(a)$	$E_1(a)$	$E_2(a)$	$X(a,b)$	$\mathrm{Cov}[X,X(a,b)]$	$\beta(a,b)$
0	100	1.0000	120.00	18000.0	87.98	702.5	0.27
100	200	0.5595	87.97	15536.5	26.90	1903.9	2.36
200	300	0.0938	23.88	6383.6	4.24	726.6	5.71
300	400	0.0148	5.31	1964.4	0.70	194.2	9.25
400	500	0.0027	1.24	591.7	0.13	51.6	12.83
500	Infinity	0.0006	0.32	186.7	0.04	21.3	17.74
0	Infinity				120.00	3600.0	1.00

Notice that the layer betas are all lower with the risk load than without it. This is intuitively plausible, since, as discussed in Appendix 4, the LPS transforms lower layers into higher ones. The transformation is in proportion to $1+\lambda = 1.20$. For example, $G(100) = 0.4066 = \hat{G}(120)$ and $G(200) = 0.0442 = \hat{G}(240)$. Thus, the risk-neutral probability transformation (location parameter shift) translates the layer from 100 to 200 into the layer from 120 = 100(1.20) to 240 = 200(1.20).

Finally, we calculate the layer betas using the 0.20 overall risk load and the fractional PH transform with $m = 0$ and $q = 0.7102$, shown in table 3.3. The partial moments $E_1(x)$ and $E_2(x)$ are obtained by numerical integration. $E_0(x) = \hat{G}(x)$ is obtained directly from the fractional PH transform.

Table 3.3

Calculation of Layer Betas for Numerical Example
$\lambda = 0.20$; *Point Risk Load by Fractional PH Transform with* $m = 0$

a	b	$E_0(a)$	$E_1(a)$	$E_2(a)$	$X(a,b)$	$Cov[X,X(a,b)]$	$\beta(a,b)$
0	100	1.0000	120.00	19144.9	86.27	869.8	0.25
100	200	0.4422	77.95	15717.7	26.49	2220.7	2.12
200	300	0.0903	25.30	7584.2	5.46	1037.3	4.80
300	400	0.0201	7.81	3174.8	1.27	380.5	7.57
400	500	0.0052	2.60	1336.6	0.34	140.6	10.35
500	Infinity	0.0016	0.94	586.1	0.16	96.4	14.80
0	Infinity				120.00	4745.2	1.00

Here, compared to the LPS transform of Table 3.2, the layer betas are lower for all layers, but nearly the same for the layers 0 to 100 and 100 to 200. However, the betas at the high layers do not change by much compared to the PRL. As observed above, the ratio of the PRL values at $x = 1000$ is 14.9, a 1390% change. The corresponding ratio of point betas is 0.79, only a 21% change.

Section 4: Parameter Measurement and Pricing Method

The theory in the preceding sections has little practical value unless we can estimate the parameter values and apply them in a realistic pricing exercise. This section uses historical data to provide the parameter estimates and also expands the pricing model (introduced in Section 1) so that it can be used in a catastrophe reinsurance application.

Since the theory uses market valuation, data using statutory accounting principles must be converted to an economic basis. For example, the pricing model discounts cash flows to give market values. Appendix 5 contains the relevant exhibits for this section.

My primary intent is to illustrate the capital allocation method as applied to reinsurance. To disguise proprietary information and to provide an incentive for others to do further research in this area, I have rounded some values that can be estimated more precisely. For a more important application, such as a rate filing, the parameter selection should be done more carefully. However, the results determined here are in the right ballpark and appear to be consistent with observed reinsurance prices.

The pricing example in Section 5 sets a fair premium for coverage of losses occurring during 1999, so I have used parameter estimates based on data extrapolated to that period.

Constructing the Representative Insurer

To build a model that will approximate a market price for insurance coverage, we need to examine a typical insurer within the industry. We call this abstraction the *representative insurer*. For most of the parameters needed for the representative insurer, we use industry data and assume that the representative insurer is a proportionate scaled-down version of the industry. However, some parameters need to be adjusted further.

For example, if the industry were a single monopolistic insurer, far less capital would be required to provide the same default ratio than for the representative insurer. The representative insurer could not be as diversified as the entire industry.

Estimating the Overall Capital Ratio

Exhibit 1 uses 1991 to 1995 consolidated industry data to estimate the overall capital ratio for the average insurer. Key assumptions in this analysis are that the ratio of statutory surplus to GAAP equity is 0.800 and that the economic value of the loss and LAE reserves is a 12% discount from the recorded statutory values (the discounting is shown in Exhibit 2). The indicated capital ratio is 0.610.

However, if the industry has inadequate loss reserves, then this value will be lower. If we use a more recent time period, the capital ratio will be higher. The issue remains as to whether the actual amount of capital held by the industry is a normative or equilibrium value. Some analysts would argue that currently the industry is over-capitalized.

I believe that a capital ratio around 0.50 is a reasonable first approximation.

Estimating the Average Loss CV

Exhibit 3 uses 1982 to 1997 industry loss and LAE reserve data to estimate the annual loss CV of liability (including Workers' Compensation) lines at 0.059. The corresponding non-catastrophe loss CV is 0.044. The composite value is 0.053. But, due to maximum diversification, the consolidated industry loss CV is undoubtedly lower than that of the typical insurer. Therefore, the loss CV estimate should most likely exceed 0.08. For the purpose of approximating the capital allocation to lines of business, I have used $\sigma_L = 0.100$.

Estimating the Catastrophe Loss CV

Cummins, Lewis and Phillips (1996) find that the lognormal distribution fits individual large industry catastrophe losses fairly well and that the CV ranges between 2.1 and 2.3. Since the individual losses can be assumed to be statistically independent, the CV of the aggregate losses is lower than that for an individual loss.[15] However, the catastrophe loss CV for the representative insurer would be larger than that for the industry, due to inefficient diversification.

RMS (Risk Management Solutions) engineering simulation modeling estimates of individual insurer aggregate catastrophe losses show that the CV is in the upper region of the above range and that the lognormal distribution fits the simulated annual aggregate losses fairly well. Thus, I have selected the annual aggregate catastrophe loss CV as 2.30.

Estimating the Asset CV

Since investment returns for most insurers arise from well-diversified portfolios, the average historical asset CV values should adequately represent those of a typical insurer. Exhibit 1 and Exhibit 4 use 1945 to 1995 data to produce an industry-average asset CV of 0.075. In this analysis, I adjusted the bond CV to represent that of a typical property-liability insurer (having a shorter average bond duration than the source data). I also weighted the CV values by asset class to represent the average insurer's portfolio mix.

Estimating the Capital Allocation Factor Z

From above, we have $\sigma_A = 0.075$ and $\sigma_L = 0.100$, giving a composite CV of $\sigma = 0.125$. Since the overall capital ratio is $c = 0.500$, the formula for the definition of Z in equation (1.18) gives Z = 0.427.

Estimating the Overall Catastrophe Risk Load

Here we run into a brick wall. It is clear that catastrophe insurance requires a risk load and that high-layer catastrophe reinsurance requires a very large risk load. However, I have not found any good empirical data to analyze in order to estimate what loading the market charges for catastrophe risk.

[15] Let $\tilde{Y} = \sum \tilde{X}_i$ be the sum of n independent identically distributed random variables $\tilde{X}_i$. The CV of $\tilde{Y}$ is $\mathrm{CV}(\tilde{Y}) = \sqrt{\mathrm{Var}(\tilde{Y})} / \mathrm{E}(\tilde{Y}) = \sqrt{n\,\mathrm{Var}(\tilde{X}_i)} / [n\mathrm{E}(\tilde{X}_i)] = \mathrm{CV}(\tilde{X}_i)/\sqrt{n}$.

One possible avenue is to examine rates for property insurance, filed with state insurance departments. However, the actual approved rates will probably not represent unbiased, market-clearing prices, since in many states insurers are not free to charge what they wish for property insurance, especially in politically sensitive personal lines.

Another avenue is to analyze prices for catastrophe reinsurance and extrapolate the implied risk load to the entire range of losses. Here, we may be able to obtain market prices for a handful of companies, but even that amount of data is difficult to get, due to the reluctance of both ceding insurers and assuming reinsurers to provide it. Even with this data, the embedded risk load must be separated from the expected loss and the tax burden of the capital requirement. Another issue is that what appears to be a huge risk load may partially reflect the illiquid nature of high-layer catastrophe reinsurance. Few, if any reinsurers are willing to take on that much risk.[16]

In view of the above discussion, I have chosen a 0.50 overall risk load for illustrating the capital allocation and pricing method. This value is little more than a guess. Also, I have initially assumed the LPS transform to establish risk loads by layer and the corresponding capital amounts by layer (the PH transform is also considered in Section 5).

Interest Rate and Income Tax Rate

The certainty-equivalent loss is discounted to present value using a default-free interest rate of 6%. This value is higher than the current one-year U. S. Treasury yield rate, but well within the range of recent experience.

We assume a 35% Federal income tax rate. This value is the current corporate tax rate.

[16] It will be interesting to see if non-insurance risk financing mechanisms such as catastrophe bonds and catastrophe futures will grow enough to offer material coverage that the reinsurers are currently unwilling to provide. If so, their prices will give another method to infer risk loads.

Estimating the Catastrophe Parameters

From Exhibit 1 (Appendix 5) we have the industry total market value of liabilities in 1995 at $499 billion. We extrapolate the market value of liabilities to $550 billion at the beginning of 1999. Based on projecting expected individual company losses to the industry level using market share data, the expected annual industry catastrophe losses for 1999 are roughly $5 billion. To convert to a market value, we multiply by 1 plus the overall risk load and take the present value using the risk-free interest rate. This result is $7.08 billion = 5(1.5)/(1.06). Thus, the weight of market value of catastrophe liabilities to all liabilities is 0.0129 = 7.08/550.

From Section 2, we use the relationship $\beta_i = w_i(\sigma_i^2 / \sigma_L^2)$ to get the loss beta for catastrophes, considered as a line of business. Consequently, the overall catastrophe beta is $\beta_k = 0.0129[2.30^2]/[0.100]^2 = 6.81$. Thus the capital ratio for catastrophes is $c_k = c + (\beta_k - 1)Z$, or $c_k = 2.98 = 0.500 + (6.81 - 1)(0.427)$. The capital allocation factor for catastrophes is $Z_k = 2.91 = 6.81(0.427)$.

Pricing Method

To estimate fair reinsurance prices, we use the present value model of equation (1.1), except with no default. Estimating the expected default of reinsurers is outside the scope of this paper.[17]

We assume that premium is collected at the effective date of the coverage and there are no administrative expenses. Loss adjustment expenses are included in the losses. The loss is expected to occur at the middle of the exposure period and the payment is made 0.5 years later. Thus, the loss payment happens one year from the contract effective date. This payment pattern should be a reasonable approximation to the actual payment of

[17] For high-layer catastrophe reinsurance, there is a good possibility that losses large enough to pierce the coverage layer will bankrupt some property reinsurers. Thus the default expectation is not trivial and can be considered a hidden, additional cost of the reinsurance. The default is extremely difficult to price, since several reinsurers usually participate in the treaty. Often they are foreign, with limited data from which to assess their solvency prospects.

catastrophe losses (a more precise estimate could incorporate the seasonality of catastrophe losses).

Since the losses are paid at the end of one year, the fair premium $P(a,b)$ for the layer from a to b is a fairly simple function of the expected loss in the layer, its risk load and the required capital. Appendix 6 develops the fair premium formula

$$(4.1) \quad P(a,b) = \frac{X(a,b)[1+\lambda(a,b)]}{1+r} + \frac{X(a,b)[1+\lambda(a,b)]\,c(a,b)\,r\,t}{(1+r)(1-t)}.$$

Notice that the fair premium can be separated into three components:

(1) the present value of the expected loss, or $X(a,b)/(1+r)$,

(2) the present value of the risk load, or $X(a,b)\,\lambda(a,b)/(1+r)$ and

(3) the present value of the capital cost, or $\frac{X(a,b)[1+\lambda(a,b)]\;c(a,b)\,rt}{(1+r)(1-t)}$.

For comparison between layers, it may be useful to calculate the expected loss ratio implied by the fair premium. The expected loss ratio for the layer is $ELR(a,b) = X(a,b)/P(a,b)$.

Since Section 2 developed the capital allocated to a layer (a to b) and Section 3 the risk load applied to the layer, it is straightforward to get the expected return on the capital (equity) allocated. Appendix 6 shows that the return on equity (ROE) for the layer is

$$(4.2) \quad R(a,b) = r+(1-t)\frac{1+r}{1+\lambda}\frac{\lambda(a,b)}{c(a,b)} = r+(1-t)\left[\frac{1+r}{1+\lambda}\right]\frac{\lambda(a,b)}{c_k+[\beta(a,b)-1]Z_k}.$$

Section 5: Catastrophe Reinsurance Application

The example here is chosen to be fairly realistic, but it is simplified to eliminate some details that inevitably surround a real pricing exercise. For example, administrative expenses are assumed to be zero.

The problem is to obtain a fair price for high-layer catastrophe reinsurance protection for a medium to large size insurer with $50 million of annual expected losses for natural property catastrophes. The reinsurance contract will cover total aggregate losses for all catastrophes occurring during 1999.

Because there is *no a priori* reason to expect that losses from natural catastrophes are correlated with losses from other lines, we can use equation (2.6) to estimate the capital for a layer. Thus, the calculation is greatly simplified by ignoring the covariance between the layer loss and losses from other lines. Also, since many insurers use extensive catastrophe models, the parameters of the catastrophe loss distribution can be readily estimated. A final simplifying element is that the cash flows from catastrophe coverage are short-duration. Consequently, what might appear initially as a very difficult pricing problem is actually much easier to solve.

The resulting fair price estimate can be used in negotiations with reinsurers over the price of a desired contract or for deciding on how much coverage to buy (how high the layer). The estimates may help indicate which layers are under- or over-priced relative to each other. Another use for the estimate is to document the cost of coverage between a ceding U.S. insurer and an assuming foreign affiliate. If the Treasury Department believes that the cost is excessive, it may disallow a portion of the reinsurance premium.

Using the Parameters to Get a Layer Capital Formula

From Section 4, we have $Z_k = 2.91$ and $c_k = 2.98$. So the capital allocation formula, based on equation (2.6), becomes

(5.1) $c(a,b) = 2.98 + [\beta(a,b) - 1][2.91]$.

Capital Ratio and Risk Load by Layer

We examine all layers up to $1 billion in $100 million increments. Using equation (5.1), we get the capital ratio for each layer. The capital amount is the capital ratio times the market value of the loss, or $C(a,b) = c(a,b)\hat{X}(a,b)/(1+r)$. The risk load is derived by the method used in Figure 3.1. Table 5.1 shows the right-tail probability, expected loss, certainty-equivalent expected loss, layer beta, capital ratio, risk load and capital amount for each layer:

Table 5.1

Capital Ratio, Risk Load and Capital Amount by Layer for Catastrophe Pricing Example ($Millions)

a	b	$E_0(a)$	$X(a,b)$	$\hat{X}(a,b)$	$\beta(a,b)$	$c(a,b)$	$\lambda(a,b)$	$C(a,b)$
0	100	1.0000	33.40	42.72	0.18	0.60	0.279	24.38
100	200	0.1867	7.24	12.25	0.69	2.08	0.693	24.09
200	300	0.0806	3.19	6.00	1.13	3.36	0.877	19.03
300	400	0.0445	1.76	3.53	1.55	4.58	1.008	15.24
400	500	0.0279	1.09	2.30	1.95	5.75	1.111	12.48
500	600	0.0189	0.73	1.60	2.35	6.91	1.198	10.42
600	700	0.0135	0.51	1.16	2.74	8.04	1.273	8.83
700	800	0.0100	0.38	0.88	3.13	9.16	1.340	7.59
800	900	0.0077	0.28	0.68	3.51	10.27	1.400	6.60
900	1000	0.0060	0.22	0.54	3.89	11.36	1.455	5.79
1000	Infinity	0.0048	1.20	3.35	8.29	24.17	1.790	76.33
0	Infinity		50.00	75.00	1.00	2.98	0.500	210.78

Notice that the capital allocated to layers above $500 million (four standard deviations above the mean) is $115.6 million—over half the total. The $75 million certainty-equivalent expected loss equals the $50 million pure expected loss increased by the overall 0.50 risk load.

Fair Premium Estimate by Layer

Given the layer capital ratios and risk loads, equation 4.1 is used to determine the fair premium and equation 4.2 the implied ROE by layer. Table 5.2 shows the fair premium by layer, separated into its components (present value of expected loss, risk load and capital cost). Table 5.2 also provides the implied ROE by layer.

Table 5.2

Estimated Fair Premium by Component,
Expected Loss Ratio and Implied ROE by Layer
For Catastrophe Pricing Example
($Millions)

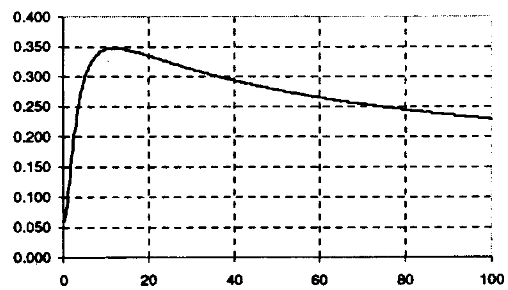
a	b	Expected Loss	Capital Cost	Risk Load	Fair Premium	Expected Loss Ratio	Implied ROE
0	100	31.51	0.74	8.78	41.04	0.814	0.272
100	200	6.83	0.73	4.73	12.29	0.589	0.213
200	300	3.01	0.58	2.64	6.24	0.512	0.180
300	400	1.66	0.46	1.67	3.79	0.463	0.161
400	500	1.03	0.38	1.14	2.55	0.427	0.149
500	600	0.69	0.32	0.82	1.83	0.398	0.140
600	700	0.48	0.27	0.62	1.37	0.374	0.133
700	800	0.35	0.23	0.47	1.06	0.354	0.127
800	900	0.27	0.20	0.37	0.84	0.336	0.123
900	1000	0.21	0.18	0.30	0.69	0.321	0.119
1000	Infinity	1.13	2.33	2.03	5.48	0.219	0.094
0	Infinity	47.17	6.42	23.58	77.18	0.648	0.137

Results from this example seem to agree in magnitude with observed market prices. The expected loss ratios are extremely low for high layers. Notice that the ROE by layer *decreases* with increasingly high layers. This occurs because the risk load does not increase as fast as the capital allocated to the layer. The overall implied ROE of 13.7% can be found using equation (4.2) (with $a = 0$ and b infinite). Since $\lambda = 0.50$ and $c_k = 2.98$, the overall ROE is $0.137 = 0.060 + (1-0.35)(1.036)(0.50)/[(1.5)(2.98)]$.

Although it appears from Table 5.2 that the implied ROE is highest at the lowest layer, it is not. As observed in Appendix 6, the (point) ROE for the thin layer at zero is equal to the risk-free rate, which is 6% here. Figure 5.1 shows the ROE for point layers up to $100 million.

Figure 5.1

Point ROE for Catastrophe Pricing Example



Loss Size

As seen above, the implied ROE peaks at about 35% for a narrow layer centered on approximately $12 million.

Because the shape of the point ROE curve is non-linear, with a minimum value of r, it is not feasible to use a pricing model that has a *single* required ROE as an input.

Results with Different Risk Loads

To see the impact of a change in the intensity of risk loads at high layers, we use the fractional PH transform of equation (3.6) with $m = 0$ and $q = 0.813$. These parameters maintain the overall risk load at 0.50, but increase the risk load at higher layers. Table 5.3 compares the capital amounts, risk load amounts, and fair premiums by layer for the PH transform to those of the above LPS transform.

Table 5.3

Comparison of Fair Premium Components by Layer,
LPS vs. PH Transform
For Numerical Example
($Millions)

		PH Transform			LPS Transform		
u	*b*	Capital Amount	Risk Load	Fair Premium	Capital Amount	Risk Load	Fair Premium
0	100	19.17	6.16	38.25	24.38	8.78	41.04
100	200	18.61	4.26	11.66	24.09	4.73	12.29
200	300	15.23	2.70	6.18	19.03	2.64	6.24
300	400	12.65	1.86	3.91	15.24	1.67	3.79
400	500	10.71	1.36	2.71	12.48	1.14	2.55
500	600	9.23	1.03	2.00	10.42	0.82	1.83
600	700	8.05	0.81	1.54	8.83	0.62	1.37
700	800	7.11	0.65	1.22	7.59	0.47	1.06
800	900	6.33	0.53	0.99	6.60	0.37	0.84
900	1000	5.68	0.44	0.82	5.79	0.30	0.69
1000	Infinity	98.01	3.77	7.89	76.33	2.03	5.48
0	Infinity	210.78	23.58	77.18	210.78	23.58	77.18

Compared to the LPS transform, the PH transform has both greater capital requirements and risk loads at high layers. Since the overall capital requirement and risk loads are the same, the PH transform has both smaller capital requirements and risk loads at low layers. However, notice that, for the two methods, the amount of capital allocated to layers is closer to being the same than is the risk load.

Section 6: Discussion of Results

The results of each section are summarized in the Introduction. This section discusses the implications of the results.

The Myers-Read Model and Loss Beta

The MR method is derived from underlying economic assumptions that govern the behavior of financial markets. Not surprisingly, the resulting capital allocation is such that the insurer's total capital equals the sum of the capital amounts allocated to the component policies.

The *loss beta* is the important attribute of a policy that determines the amount of capital allocated to it. A policy will have a high/low beta if it has a high/low loss CV or if it has a high/low correlation with other policies. A high-CV policy or line of business may not require a great amount of capital if it has a low correlation with all other lines of business.

Relationship between Risk Loads and Capital Allocation

Since capital allocation depends on the market values of losses by layer, not just the expected losses, the underlying risk load process influences the economic basis for capital allocation to layer. Both the allocated capital and risk loads depend on the particular loss size distribution. They both also share the value additivity property, where the sum of capital and risk loads over a subdivision of policies or layers, equals the total value.

However, the two processes are not the same. Capital allocation relies on the default risk of a typical insurer, including sources of risk that do not command a risk premium in financial markets. The risk loads for losses depend on non-diversifiable risk and do not contain any default component. For example, a typical insurer has some specific risk that would be diversified away if the insurer represented a proportional share of the entire industry. Also, the asset risk of a representative insurer will affect capital allocation, but will not influence the risk load. This happens because the insurer's assets are already fairly priced by themselves. With risk-free assets, the risk load for losses would be the same as with risky assets.

Role of Capital Allocation in Setting Premiums

For an insurer, the cost of carrying capital is the present value of the income taxes on investment income from the capital. If there were no corporate income taxes, the capital cost would be zero and the allocation of capital to line or policy would be unnecessary for determining premiums. However, the market would still demand a risk load and the premium would exceed the present value of the expected losses and expenses.

The expected return on capital by layer can be determined from the market or fair premium, which depends on the capital allocation and risk load by layer. Because the amount of capital and the risk load arise from different economic processes, the expected ROE will not be a constant across all layers. In fact, the lowest layer will produce an expected ROE equal to the riskless rate.

Consequently, one cannot use a constant expected ROE in establishing a fair premium by layer. By extension, this argument is also true for individual policies. At the policy level, a more fundamental problem is that it is possible for the allocated capital for a low-risk policy to be negative. Thus, the role of capital in pricing cannot be as the denominator in a return on equity calculation. Rather, it is a basis for determining the value of income tax costs.

Although ROE as a profit measurement may be useful at the level of the entire insurance company, we conclude here that ROE is an inadequate and potentially misleading profitability measure at the line of business level and below. For determining fair prices for lines, policies and individual layers of insurance, a present value method is more appropriate.

Catastrophe Reinsurance Pricing

I have demonstrated how parameter estimates for the MR model can be developed from industry data. Because catastrophe losses are assumed to be uncorrelated with losses from other lines, the capital allocation method is greatly simplified. Given some basic assumptions regarding the distribution of catastrophe losses and risk load behavior, I have shown that the resulting capital allocation produces reasonable-looking market premiums for reinsurance protection.

A major unresolved issue is the nature of the risk load process for catastrophe losses. Admittedly there is no economic justification for any particular risk load process (except perhaps the location parameter shift). However, I have shown that the extended MR capital allocation method can accommodate any probability transformation that produces positive risk loads by layer.

Areas for Future Research

Since the risk load influences capital allocation as well as being a separate component of the premium, it is extremely important in estimating fair premiums. For catastrophe losses, since the overall risk load is apparently so large, its importance increases. Unfortunately, we know very little about the economic basis for catastrophe risk loads. This area requires extensive investigation.

The parameters used in estimating the capital allocation factor Z were extrapolated from industry-level data in order to characterize the representative insurer. These estimates can be improved by analyzing individual insurers and averaging the results.

Conclusion

This paper has explained the development of the Myers-Read capital allocation model and has extended it for practical insurance applications. An important use is catastrophe reinsurance pricing, which many consider to be a difficult problem. The results shown in the above exercise reveal the power of this new method.

Glossary of Notation

The *location* below indicates the equation number nearest to where the symbol or acronym first appears. F denotes a footnote (if applicable).

Symbol	*Location*	*Definition*
a	2.1	Lower limit of a coverage layer
$\tilde{a}$	1.10	Normalized asset (random) value
a_0, a_1	A4.6	Constants in the approximation for the point risk load
A	1.1	Market value of assets for entire insurer (at beginning of period)
A'	1.1	Market value of assets for entire insurer at the end of the period
b	2.1	Upper limit of a coverage layer
c	1.1	Overall capital ratio for all lines: capital per unit of loss value
c_i	1.1	Capital ratio for line or policy i
c_k	1.19	Capital ratio for line containing policy i
$c(a,b)$	2.6	Capital ratio for layer from a to b
C	1.1	Capital amount for the entire insurer
C_i	1.1	Capital amount for line or policy i
$C(a,b)$	5.1	Capital amount allocated to the layer from a to b
CV	1.4	Coefficient of variation
d	1.1	Market value of expected default per unit of expected loss for entire insurer
d_i	1.1	Market value of expected default per unit of expected loss for line/policy i
D	1.1	Market value of expected default for entire insurer
D_i	1.1	Default value for policy or line i
$ELR(a,b)$	4.2	Expected loss ratio for the layer from a to b
$E_n(u)$	2.2	Partial nth moment evaluated at u
$f(x)$	3.1	Probability of loss amount x
$\hat{f}(x)$	3.1	Risk-neutral probability of loss amount x
$F(u)$	2.2	Cumulative probability distribution, evaluated at u
$G(u)$	2.7	Right tail of cumulative probability distribution, evaluated at u
$\hat{G}(u)$	3.3	Right tail of cumulative risk-neutral probability distribution, evaluated at u
GBM	3.5	Geometric Brownian motion
h_i	1.1	Default allocation constant for line/policy i
$h(x)$	A2.2	Function that determines coverage at loss value x
k	A4.7	Constant in translation of the volatility
k_E	A3.5	Parameter of exponential distribution

k_P	A3.4	Parameter of Pareto distribution
κ	F18	Parameter of alternative Pareto distribution
$\tilde{\ell}$	1.10	Normalized liability value for all losses
$\tilde{\ell}_i$	1.10	Normalized liability value for line/policy *i*
L	1.1	Market value of losses for entire insurer (at beginning of period)
L'	1.1	Market value of losses for entire insurer at the end of the period
L_i	1.1	Market value of loss for line/policy *i*
LPS	3.6	Location parameter shift
m	3.6	Parameter in fractional PH transform
MR		Myers-Read (model)
$MV(a,b)$	A6.1	Market value of the expected loss in the layer from *a* to *b*
MVL	A6.3	Market value of the expected loss for the insurer
n	2.2	Index in partial moment definition
$n(u)$	1.8	Standard normal density function evaluated at *u*
N_1, N_2	1.6	Normal variables in equation (1.6)
$N(u)$	1.6	Standard normal cumulative distribution function evaluated at *u*
P	1.1	Fair premium for all policies of an insurer
P_i	1.1	Fair premium for line or policy *i*
$P(a,b)$	4.1	Fair premium for layer from *a* to *b*
PH	3.4	Proportional hazard (transform)
PRL	3.3	Point risk load
PVC	A6.4	Present value of the income taxes from capital
q	3.4	Parameter in basic PH transform
$q(x)$	3.4	Generalized PH transform at loss value *x*
r	F5	Risk-free interest rate
r_A	F2	Investment return
$\dot{r}$	A4.4	Exponential form of risk-free interest rate
R	A4.4	Expected stock return
$\dot{R}$	A4.4	Exponential form of expected stock return
$R(a,b)$	4.2	Return on equity for the layer from *a* to *b*
$R(x)$	A6.8	Point ROE evaluated at loss size *x*
ROE	4.2	Return on equity
s	2.7	CV of policy loss
t	F2	Income tax rate
T	1.1	Income tax cost per unit of capital
TP	A6.3	Income taxes paid
w_i	1.10	Weight of losses from line/policy *i* compared to total losses
$w(a,b)$	2.5	Weight of losses from layer compared to total policy loss
x^*	A4.8	Value of loss at which the point risk load becomes negative
X	2.1	Expected value of individual policy loss
$\tilde{X}$	2.3	Value of individual policy loss

$\hat{X}$	3.1	Certainty equivalent loss value
$X(a,b)$	2.3	Expected value of loss in the layer from a to b
$\tilde{X}(a,b)$	2.1	Value of loss in the layer from a to b
$\hat{X}(a,b)$	3.2	Certainty equivalent loss in the layer from a to b
y	1.6	A variable in equation (1.6)
Y	1.4	Mean of lognormal distribution
Z	1.18	Capital allocation factor
Z_k	2.6	Within-line capital allocation factor
α	A3.3 A3.4	Parameter of gamma distribution Parameter of Pareto distribution
β_i	1.17	Loss beta for line/policy i relative to all losses
β_k	1.19	Loss beta for line k relative to all losses
β_{ik}	1.19	Loss beta for line/policy i relative to line k containing it
$\beta(a,b)$	2.5	Loss beta in layer from a to b, relative to entire policy.
$\beta(x)$	2.7	Point beta at loss size x
δ	A4.7	Amount of location parameter shift
Δ	2.7	Width of narrow layer
γ	A3.3	Parameter of gamma distribution
γ_i	1.17	Ratio of covariance of policy loss and assets to that of all losses and assets
$\Gamma(\alpha;\gamma x)$	A3.3	Gamma distribution cumulative probability evaluated at x
λ	3.1	Risk load for entire loss distribution
$\dot{\lambda}$	3.5	Exponential form of risk load for entire loss distribution
$\lambda(x)$	3.3	Point risk load evaluated at loss size x
$\dot{\lambda}(x)$	A4.5	Exponential form of point risk load evaluated at loss size x
$\lambda(a,b)$	3.2	Risk load in the layer from a to b
μ	1.4 A3.2	Location parameter in lognormal distribution Mean of normal distribution
μ'	3.5	Shifted lognormal location parameter
μ_a	1.10	Location parameter for normalized asset value distribution
μ_ℓ	1.10	Location parameter for normalized liability value distribution
ν	1.4 1.5	Dispersion parameter in lognormal distribution; volatility Combined asset and loss volatility in equation (1.5)
ν_A	1.5	Asset volatility
ν_L	1.5	Loss volatility
ν_{AL}	1.5	Covariance (assets and losses) parameter
6	A3.5	Parameter of exponential distribution
ρ_i	1.19	Correlation between policy i losses and all losses
ρ_k	1.19	Correlation between line k losses and all losses

ρ_{ik}	1.19	Correlation between policy *i* losses and line *k* losses
ρ_{ij}	1.10	Correlation between policy *i* and policy *j* losses
ρ_{iL}	1.17	Correlation between policy *i* losses and all losses
σ	1.10	Composite loss and asset CV for entire insurer
σ_A	1.10	Asset CV for insurer
σ_L	1.10	Loss CV for insurer
σ_{AL}	1.10	Covariance between normalized asset and liability values
σ_i	1.10	Loss CV for policy or line *i*
σ_{iA}	1.15	Covariance of loss CV with the asset CV
σ_{iL}	1.15	Covariance of loss CV for policy *i* with CV for all losses
ω_i	1.12	Correlation coefficient between normalized asset value and normalized liability value for line/policy *i*

References

Allen, Franklin, J. David Cummins and Richard D. Phillips, 1996, Financial Pricing of Insurance in a Multiple Line Insurance Company, [Center for Risk management and Insurance Research, Georgia State University, Atlanta, GA], Working Paper no. 95-2.

Black, Fischer and Myron Scholes, 1973, The pricing of Options and Corporate Liabilities, *Journal of Political Economy*, May-June, 81: 637-659.

Brealy, Richard A. and Stuart C. Myers, 1996, *Principles of Corporate Finance* (5th Edition) McGraw-Hill, New York.

Butsic, Robert P., 1994, Solvency Measurement for Risk-Based Capital Applications, *Journal of Risk and Insurance*, 61: 656-690.

Cox, John C. and Mark Rubinstein, 1985, *Options Markets* Prentice-Hall, Englewood Cliffs.

Cummins, J. David, 1988, Risk-Based Premiums for Insurance Guaranty Funds, *Journal of Finance*, September, 43: 823-838.

Cummins, J. David, Christopher M. Lewis and Richard D. Phillips (1996), Pricing Excess-of Loss Reinsurance Contracts Against Catastrophic Loss, [Center for Risk management and Insurance Research, Georgia State University, Atlanta, GA], Working Paper No. 95-7.

Derrig, Richard A., 1989, Solvency Levels and Risk Loadings Appropriate for Fully Guaranteed Property-Liability Insurance Contracts: A Financial View, *Financial Models of Insurance Solvency*, J. D. Cummins and R. A. Derrig eds., Kluwer Academic Publishers, Boston, 303-354.

Derrig, Richard A., 1994, Theoretical Considerations of the Effect of Federal Income Taxes on Investment Income in Property-Liability Ratemaking, *Journal of Risk and Insurance*, 61: 691-709.

Doherty, Neil A. and James R. Garven, 1986, Price Regulation in Property-Liability Insurance: A Contingent-Claims Approach, *Journal of Finance*, December, 41: 1031-1050.

Feller, William G, 1968, *An Introduction to Probability Theory and Applications*, John Wiley & Sons, New York.

Meyers, Glenn G., 1994, The Competitive Market Equilibrium Risk Load Formula for Increased Limits Ratemaking, *Proceedings of the Casualty Actuarial Society*, LXXVIII.

Miccolis, Robert S. 1977, On the Theory of Increased Limits and Excess of Loss Pricing, *Proceedings of the Casualty Actuarial Society*, LXIV.

Myers, S. C. and R. Cohn, 1987, Insurance rate of Return Regulation and the Capital Asset Pricing Model, *Fair Rate of Return in Property Liability Insurance*, in J. D. Cummins and S. Harrington, eds. Kluwer-Nijhoff Publishing Company, Norwell MA.

Myers, S. C. and J. Read, 1998, Line-by-Line Surplus Requirements for Insurance Companies. [Unpublished paper originally prepared for the Automobile Insurance Bureau of Massachusetts.]

Panjer, Harry H., (editor), 1998, *Financial Economics*, The Actuarial Foundation, Schaumburg, IL.

Venter, Gary G., 1991, Premium Implications of Reinsurance Without Arbitrage, ASTIN Bulletin, 21 No. 2: 223-232.

Wang, Shaun, 1998, Implementation of the PH-Transform in Ratemaking, [presented at the Fall, 1998 meeting of the Casualty Actuarial Society].

Appendix 1

Separating Line and Policy Correlation Coefficients

Equation (1.18) applies both to a line (indexed by *k*) and a policy (indexed by *i*), so

(A1.1) $$c_k = c + (\beta_k - 1)Z\text{, and}$$

(A1.2) $$c_i = c + (\beta_i - 1)Z.$$

Here, $\beta_k = \rho_k \sigma_k / \sigma$ and $\beta_i = \rho_i \sigma_i / \sigma$, where β_k is the loss beta for the line relative to all lines, β_i is the loss beta for the policy relative to all lines, ρ_k is the correlation between the line losses and losses of all lines (including line *k*) and ρ_i is the correlation between the policy losses and losses of all lines (including line *k*). Also, σ_i, σ_k and σ are the respective loss CV values for the policy, line and total of all lines.

Eliminating the variable *c* in equations (A1.1) and (A1.2), we get

(A1.3) $$c_i = c_k + (\beta_i - \beta_k)Z \;=\; c_k + (\beta_i / \beta_k - 1)\beta_k Z.$$

Define $\beta_{ik} = \rho_{ik} \sigma_i / \sigma_k$ as the loss beta for the policy relative to the line containing it. Here, ρ_{ik} is the correlation between the policy losses and losses of the line *k*. From the above definitions of β_k and β_i, we have

(A1.4) $$\beta_i / \beta_k = (\rho_i / \rho_k)(\sigma_i / \sigma_k) = (\rho_i / \rho_k)(\beta_{ik} / \rho_{ik}) = (\rho_i \beta_{ik})/(\rho_k \rho_{ik}).$$

Substituting equation (A1.4) into equation (A1.3), we get

(A1.5) $$c_i = c_k + \left(\frac{\rho_i}{\rho_{ik}\rho_k} \beta_{ik} - 1 \right) \beta_k Z.$$

If the covariance of losses in line *k* with all other losses (excluding line *k*) is zero, then $\beta_i = w_i\sigma_i / \sigma$ and $\beta_k = w_k\sigma_k / \sigma$, where w_i and w_k are the respective policy and line weights for loss values relative to the total losses of all lines. Thus,

(A1.6) $$\beta_i / \beta_k = (w_i / w_k)(\sigma_i / \sigma_k) = w_{ik}\sigma_i / \sigma_k = \beta_{ik}.$$

Notice that $w_{ik} = w_i / w_k$ is the weight of the policy losses relative to the line losses. Substituting equation (A1.6) into equation (A1.3) yields

(A1.7) $$c_i = c_k + (\beta_{ik} - 1)\beta_k Z.$$

Appendix 2

Development of Covariance Between the Layer and the Entire Loss

The covariance between two random variables is the expected value of their product minus the product of their expected values. Thus,

(A2.1) $$Cov[\tilde{X}, \tilde{X}(a,b)] = E[\tilde{X}\, \tilde{X}(a,b)] - E[\tilde{X}]\, E[\tilde{X}(a,b)].$$

We already know that $E[\tilde{X}] = X$ and that $E[\tilde{X}(a,b)] = X(a,b)$ $= E_1(a) - E_1(b) - [aE_0(a) - bE_0(a)]$ from equation (2.3). To determine $E[\tilde{X}\, \tilde{X}(a,b)]$, we define a function $h(x)$ that determines the coverage at each value of loss x. We have

(A2.2) $$h(x) = \begin{Bmatrix} 0 & & x \le a \\ x-a & for & a \le x \le b \\ b-a & & x \ge b \end{Bmatrix}.$$

Therefore,

$$E[\tilde{X}\, \tilde{X}(a,b)] = \int_0^\infty x\, h(x)\, f(x)\, dx = \int_0^a x[0]\, f(x)\, dx + \int_a^b x[x-a]\, f(x)\, dx + \int_b^\infty x[b-a]\, f(x)\, dx$$

$$= 0 + \int_a^b [x^2 - ax]\, f(x)\, dx + (b-a)E_1(b) = E_2(a) - E_2(b) - [aE_1(a) - aE_1(b)] + (b-a)E_1(b).$$

This expression reduces to

(A2.3) $$E[\tilde{X}\, \tilde{X}(a,b)] = E_2(a) - E_2(b) - [aE_1(a) - bE_1(b)].$$

Substituting equation (A2.3) into equation (A2.1), we get

(A2.4) $$Cov[\tilde{X}, \tilde{X}(a,b)] = E_2(a) - E_2(b) - [aE_1(a) - bE_1(b)] - X\, X(a,b).$$

Appendix 3

Development of Point Beta Results

Formula for Point Beta in Equation (2.7)

We want to find $\beta(x) = \lim_{\Delta \to 0} \beta(x, x+\Delta)$. From equations (2.4) and (2.5) we get

$$\beta(x,x+\Delta) = \frac{X}{Var(\tilde{X})} \frac{Cov[\tilde{X}(x,x+\Delta),\tilde{X}]}{X(x,x+\Delta)}$$

$$= \frac{1}{X\,s^2} \frac{E_2(x) - E_2(x+\Delta) + (x+\Delta)E_1(x+\Delta) - xE_1(x) - X\,X(x,x+\Delta)}{X(x,x+\Delta)}.$$

Here s is the CV of $\tilde{X}$. Thus,

$$\beta(x,x+\Delta) = \frac{1}{X\,s^2} \frac{\int_x^{x+\Delta} u^2 f(u)\,du + (x+\Delta)\int_{x+\Delta}^{\infty} uf(u)\,du - x\int_x^{\infty} uf(u)\,du}{\int_x^{x+\Delta} uf(u)\,du + (x+\Delta)\int_{x+\Delta}^{\infty} f(u)\,du - x\int_x^{\infty} f(u)\,du} - \frac{1}{s^2}$$

$$= \frac{1}{X\,s^2} \frac{\frac{1}{\Delta}\int_x^{x+\Delta} u^2 f(u)\,du - \frac{x}{\Delta}\int_x^{x+\Delta} uf(u)\,du + \int_{x+\Delta}^{\infty} uf(u)\,du}{\frac{1}{\Delta}\int_x^{x+\Delta} uf(u)\,du + \frac{x}{\Delta}\int_x^{x+\Delta} f(u)\,du - \int_{x+\Delta}^{\infty} f(u)\,du} - \frac{1}{s^2}$$

Because $\lim_{\Delta \to 0} \frac{1}{\Delta}\int_x^{x+\Delta} g(u)\,du = g(u)$ for any function g, the preceding expression becomes

$$\beta(x) = \lim_{\Delta \to 0} \beta(x,x+\Delta) = \frac{1}{X\,s^2} \frac{x^2 f(x) - x^2 f(x) + E_1(x)}{xf(x) - xf(x) + E_0(x)} - \frac{1}{s^2} = \frac{1}{s^2}\left[\frac{E_1(x)}{X\,E_0(x)} - 1\right].$$

Proof That the Point Beta Slope is Positive

We have $\beta'(x) = \frac{d}{dx}\beta(x) = \frac{d}{dx}\frac{1}{s^2}\left[\frac{E_1(x)}{X\,E_0(x)} - 1\right]$

$$= \frac{1}{s^2} \frac{X\,E_0(x)(d/dx)[E_1(x)] - E_1(x)(d/dx)[X\,E_0(x)]}{X^2[E_0(x)]^2}$$

$$= \frac{1}{s^2} \frac{X\,E_0(x)[-xf(x)] - E_1(x)[-Xf(x)]}{X^2[E_0(x)]^2} = \frac{Xf(x)}{s^2 X^2[E_0(x)]^2}[E_1(x) - xE_0(x)].$$

For $x > 0$, $\frac{Xf(x)}{s^2 X^2 [E_0(x)]^2} > 0$. Since $[E_1(x) - xE_0(x)] = \int_x^\infty (u - x) f(u)\,du > 0$ for all x, then $\beta'(x)$ must be greater than zero for all $x > 0$.

Point Beta Formulas for Various Probability Distributions

Here we use equation (2.8) to derive the point beta estimates. For each distribution, we need to find s (the CV), the right-tail probability $E_0(x)$ and the first partial moment $E_1(x)$. The chosen distributions are the lognormal, normal, gamma, Pareto and exponential. These distributions have two parameters that can be adjusted so that each distribution will have the same mean and CV.

The ***lognormal*** distribution and its parameters were discussed in Section 1. The square of the CV is $s^2 = \exp(v^2) - 1$. From equation (2.7), we have $E_0(x) = N([\mu - \ln(x)]/v)$ and $E_1(x) = X\,N([\mu - \ln(x)]/v + v)$. Thus,

(A3.1) $$\beta(x) = \frac{1}{\exp(v^2) - 1}\left[\frac{N([\mu - \ln(x)]/v + v)}{N([\mu - \ln(x)]/v)} - 1\right].$$

The ***normal*** distribution has parameters μ and σ. The mean is μ and the square of the CV is $s^2 = \sigma^2 / \mu^2$. Applying equation (2.2) to the normal density, we get $E_0(x) = N([\mu - x]/\sigma)$ and $E_1(x) = X\,E_0(x) + \sigma\, n([x - \mu]/\sigma)$. Therefore,

(A3.2) $$\beta(x) = \frac{\mu}{\sigma}\left(\frac{n([\mu - x]/\sigma)}{N([\mu - x]/\sigma)}\right).$$

The ***gamma*** distribution has parameters α and γ. Its cumulative distribution function is $F(x) = \Gamma(\alpha;\gamma x) = \int_0^{\gamma x} ([y^{\alpha-1} e^{-y}]/\Gamma(\alpha))dy$. The mean is $X = \alpha/\gamma$. The square of the CV is $s^2 = 1/\alpha$. Applying equation (2.2) to the gamma density, we get $E_0(x) = 1 - \Gamma(\alpha;\gamma x)$ and $E_1(x) = (\alpha/\gamma)[1 - \Gamma(\alpha + 1;\gamma x)]$. Therefore, (A3.3) $$\beta(x) = \alpha\left(\frac{1-\Gamma(\alpha+1;\gamma x)}{1-\Gamma(\alpha;\gamma x)} - 1\right).$$

The ***Pareto*** distribution[18] has parameters $k_P > 0$ and $\alpha > 1$. It is defined only for $x \geq k_P$ and its cumulative distribution function is $F(x) = 1-(k_P / x)^{\alpha}$. The mean is $X = \alpha k_P /(\alpha - 1)$ and the variance is finite only for $\alpha > 2$. The square of the CV is $s^2 = 1/[\alpha(\alpha-2)]$. We get $E_0(x) = 1 - F(x) = (k_P / x)^{\alpha}$ and, applying equation (2.2) to the Pareto density, $E_1(x) = (\alpha /[\alpha - 1])\, x\, E_0(x)$. Therefore,

(A3.4) $$\beta(x) = \alpha(\alpha-2)\left[\frac{x}{k_P} - 1\right].$$

The slope of the Pareto $\beta(x)$ is a constant, equal to $\alpha(\alpha-2)/k_P$.

A modified form of the ***Exponential*** distribution has parameters $k_E > 0$ and $6 > 0$. It is defined only for $x \geq k_E$ and its cumulative distribution function is $F(x) = 1-\exp(6[k_E - x])$. The mean is $X = k_E + 1/6$ and the variance is $1/\theta^2$. The square of the CV is $s^2 = 1/(1+k_E\theta)^2$. We get $E_0(x) = 1 - F(x) = \exp(6\,[k_E - x])$ and, applying equation (2.2) to the exponential density, $E_1(x) = (x + 1/6)\, E_0(x)$. Therefore,

(A3.5) $$\beta(x) = 6(1+k_E 6)\,(x - k_E).$$

The slope of the exponential $\beta(x)$ is a constant, equal to $6(1+k_E 6)$.

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[18] Another form of the Pareto distribution is defined as $F(x) = 1-(\kappa /[x+\kappa])^{\alpha}$, with $\kappa > 0$. However, this distribution has the undesirable property (for comparing point betas between alternative distributions) that the square of the CV is $\alpha /(\alpha - 2) > 1$. However, using the above form of the Pareto distribution, the CV can be any value greater than zero.

Appendix 4

Risk Loads Using Risk-Neutral Probability Distributions

Point Risk Load as a Function of Right-Hand Probability Tails

From Section 3, the *point risk load* is $\lambda(x) = \lim_{\Delta \to 0} \lambda(x, x+\Delta)$. We have

(A4.1)
$$1+\lambda(x) = \lim_{\Delta \to 0} \frac{\int_x^{x+\Delta}(u-x)\hat{f}(u)\,du + (\Delta)\int_{x+\Delta}^{\infty}\hat{f}(u)\,du}{\int_x^{x+\Delta}(u-x)f(u)\,du + (\Delta)\int_{x+\Delta}^{\infty}f(u)\,du}$$
$$= \lim_{\Delta \to 0} \frac{(1/\Delta)\int_x^{x+\Delta}(u-x)\hat{f}(u)\,du + \int_{x+\Delta}^{\infty}\hat{f}(u)\,du}{(1/\Delta)\int_x^{x+\Delta}(u-x)f(u)\,du + \int_{x+\Delta}^{\infty}f(u)\,du}.$$

Since $G(u) = 1 - F(u)$ for a cumulative probability function $F(u)$ and $\lim_{\Delta \to 0} \frac{1}{\Delta}\int_x^{x+\Delta} g(u)\,du = g(x)$ for any function $g(u)$, the limit in equation (A4.1) reduces to

(A4.2)
$$\lambda(x) = \frac{\int_x^{\infty}\hat{f}(u)\,du}{\int_x^{\infty}f(u)\,du} - 1 \quad = \quad \frac{\hat{G}(x)}{G(x)} - 1.$$

Equation (A4.2) has a form similar to that of the point beta in equation (2.8). Since $\hat{G}(0) = G(0) = 1$, we get $\lambda(0) = 0$. This is intuitively plausible since, for a thin layer of width Δ whose lower limit is zero, the actual coverage in the layer tends to be Δ with certainty. Therefore, no risk is present.

Because the average point risk load (over all values of x) must equal λ, there will be some value of the loss for which the point risk load will be greater than λ. However, since the maximum value of $\hat{G}(x)$ is 1, the value of the PRL is limited to $1/G(x) - 1 = F(x)/G(x)$. As x becomes large, this limitation in fact allows infinitely large point risk loads, which occur under many types of risk-neutral density transformations. In general, a desirable property of the PRL is that $\lambda(x) \geq 0$ for all x.

Under this non-negativity constraint, the allowable probability transforms are severely restricted.

For some applications, it is useful to define the risk load in an exponential form, so that $\dot{\lambda}(x) = \ln[1+\lambda(x)]$, or $\exp[\dot{\lambda}(x)] = 1+\lambda(x)$. We use the dot to represent the continuously compounded version of the annual rate. Notice that $\dot{\lambda}(x) = \ln[\hat{G}(x)/G(x)]$, and since the maximum possible value of $\hat{G}(x)$ is 1, we have $\dot{\lambda}(x) \le \ln[1/G(x)]$.

Generalized Proportional Hazard Transform

The proportional hazard (PH) transform is $\hat{G}(x) = [G(x)]^q$, with $0 < q \le 1$. Under the PH transform, the point risk load is simply $\lambda(x) = [G(x)]^{q-1} - 1 = [1/G(x)]^{1-q} - 1$. Since q is less than 1 and $[1/G(x)] \ge 1$, then $\lambda(x) > 0$. Also, as x becomes extremely large, the point risk load tends to infinity.

The PH transform can readily be generalized so that, instead of being constant, the exponent q is a function of x:

(A4.3) $$\hat{G}(x) = [G(x)]^{q(x)},$$

with $0 \le q(x) \le 1$. If $q(x) = 0$, then $\hat{G}(x)$ equals its maximum possible value of $[G(x)]^0 = 1$ and $\lambda(x)$ equals its maximum value of $1/G(x) - 1$. If $q(x) = 1$, then $\lambda(x) = 0$. Hence, by varying $q(x)$ between zero and 1, all feasible positive values of the PRL at x may be attained. In other words, *any* transformed distribution that gives a positive point risk load for each x can be generated by a suitable function $0 \le q(x) \le 1$. Alternatively, from equation (A4.3), the implied generalized PH transform for any risk-neutral distribution $\hat{G}(x)$ is $q(x) = \ln[\hat{G}(x)]/\ln[G(x)]$. Figure 3.1 shows the implied generalized PH transform for a lognormal distribution with risk loads specified by a location parameter shift (discussed next).

Risk Loads for the Lognormal Distribution by Shifting the Location Parameter

The classic Black-Scholes (1973) option-pricing model can be derived using the risk-neutral probability measure. Instantaneous price changes (relative to the current price) are assumed to drift according to geometric Brownian motion (GBM), which makes the relative changes in stock prices over any period length lognormally distributed. Let S be the current stock price, $\hat{S}$ the certainty-equivalent stock value at the end of one period, and R the expected return on the stock. Thus, $S = \hat{S}/(1+r) = [S(1+R)](1+\lambda)/(1+r)$, and $1+\lambda = (1+r)/(1+R)$.

Define $1+\lambda = e^{\dot{\lambda}}$, $1+r = e^{\dot{r}}$ and $1+R = e^{\dot{R}}$. This gives $\dot{\lambda} = -(\dot{R} - \dot{r})$, so the risk load for a stock will usually be *negative*, since the expected return on the stock will exceed the risk-free rate. This is intuitively plausible, because risky asset returns are discounted at high interest rates to reflect investors' aversion to low returns. Conversely, the risk load for a liability will usually be positive, reflecting policyholders' aversion to high losses.

In the case of loss values behaving like stock values (having GBM), the risk-neutral distribution is simply another lognormal distribution with a shifted location parameter but with the identical volatility as the actual loss distribution.[19] Denoting the shifted location parameter by μ', the risk load for the entire loss distribution is specified by

(A4.4) $$1+\lambda = \exp(\dot{\lambda}) = \frac{\hat{X}}{X} = \frac{\exp[\mu' + \frac{1}{2}v^2]}{\exp[\mu + \frac{1}{2}v^2]} = \exp(\mu' - \mu).$$

Thus, the exponential form of the risk load, $\dot{\lambda}$, equals the difference in the location parameters $\mu' - \mu$. For the lognormal distribution, $F(x) = N(y)$, where $y = [\ln(x) - \mu]/v$. For the risk-neutral version, $y' = [\ln(x) - \mu']/v = y - \dot{\lambda}/v$. Thus,

[19] See Panjer (1998), Chapter 10 for a discussion of the parameter shift.

$\hat{F}(x) = N(y') = N(y - \dot{\lambda}/v)$. For large values of u, we know[20] that $1 - N(u) \cong n(u)/u$. Therefore,

$$\text{(A4.5)} \qquad \dot{\lambda}(x) \cong \ln\left(\frac{n(y-\dot{\lambda}/v)}{n(y)}\right)\frac{y}{y-\dot{\lambda}/v} \cong \ln\left(\frac{n(y-\dot{\lambda}/v)}{n(y)}\right).$$

After a little manipulation, we find that $\ln[n(y-\dot{\lambda}/v)/n(y)]$ equals $a_1\ln(x)+a_0$, where $a_1 = \dot{\lambda}/v^2$ and $a_0 = -a_1[\mu + \frac{1}{2}\dot{\lambda}]$ are constants that depend on the overall risk load and the parameters of the non-transformed lognormal distribution. So equation (A4.5) simplifies to a linear function of ln(x):

$$\text{(A4.6)} \qquad \dot{\lambda}(x) \cong a_1 \ln(x) + a_0 .$$

Consequently, if the overall risk load is positive, a_1 is positive and the point risk load will tend to infinity as x approaches infinity.

Alternative View of the Lognormal LPS Transformation

An interesting feature of the lognormal transformation under the location parameter shift (LPS) is that, for any loss size x, we have $\hat{G}[x(1+\lambda)] = G(x)$. This result can be derived by substituting $z = y(1+\lambda)$ in $\hat{G}[x(1+\lambda)] = \int_{x(1+\lambda)}^{\infty} \hat{f}(z)\,dz$, where $\hat{f}(z)\,dz$ is the LPS transformed lognormal density, with $\mu' = \mu + \dot{\lambda}$. Thus, the transformed right-tail probability function is obtained simply by moving to the left of the actual right-tail distribution, from $x(1+\lambda)$ to x.

The risk-neutral method substitutes a different probability for the actual probability, keeping the loss values the same. Alternatively, we can exchange the actual loss value for a *different loss value*, keeping the *same probability*. Both methods can produce the same

[20] See Feller (1968), page 175.

$\hat{G}(x)$. For the LPS transformation, the loss value translation is the original value divided by the constant $1+\lambda$.

Creating a Larger Lognormal PRL for Higher Layers

An important property resulting from the GBM assumption for asset or liability values is the ability to create a continuous riskless hedge. If values drift in infinitesimal amounts, it is possible to simultaneously buy an option (or other derivative) and sell a quantity of the underlying asset or liability, so that the combined portfolio has no risk. Although the GBM assumption might apply to the valuation of liability[21] losses, it probably does not apply for property catastrophe losses. Here, the amount of loss is virtually unpredictable from one short time span to the next. The value of the catastrophe loss is revealed suddenly, and does not drift up and down in small increments. Therefore, it is impossible in principle to form a continuous hedge by buying coverage in a layer and selling shares of total catastrophe losses.

Because a riskless hedge cannot be created, it is reasonable to expect that, given the same overall risk load for a distribution, risk loads for high layers may be greater in the case of catastrophe losses compared to the case where losses follow GBM.

One direct way to obtain this high-layer risk load boost is to shift the volatility (dispersion parameter). For the lognormal distribution, we modify the volatility in addition to the location parameter. Thus, we let $\nu' = k\nu$ and $\mu' = \mu + \delta$. Since $(1+\lambda)X = \exp(\dot{\lambda})X = \exp(\dot{\lambda} + \mu + \frac{1}{2}\nu^2) = \hat{X} = \exp(\mu + \delta + \frac{1}{2}k^2\nu^2)$, we get the overall risk load in terms of the translation parameters δ and k:

(A4.7) $$\dot{\lambda} = \delta + (k^2 - 1)\nu^2.$$

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[21] One can view the market value of liability losses as being related to a claim cost inflation index, which can vary nearly continuously.

Notice that if $k = 1$, then Equation (A4.7) reduces to equation (A4.4) with $\delta = \dot{\lambda}$.

An undesirable consequence of the two-parameter transformed lognormal distribution is that the PRL will be *negative* for a range of loss values. This occurs when $G(x) > \hat{G}(x)$, or $1 - N\{[\ln(x) - \mu]/v\} > 1 - N\{[\ln(x) - \mu']/v'\}$. Equating the two sides of this inequality and solving for x, we get

(A4.8) $$x^* = X(1+\lambda)^{1/(1-k)} \exp[\tfrac{1}{2}kv^2].$$

Thus, when $\lambda > 0$, and $k \neq 1$, we have $\lambda(x) < 0$ for x between zero and x^*.

The negative risk load can be avoided entirely by using a suitable generalized PH transform $q(x)$. However, the cost of this remedy is that $\hat{f}(x)$ and its moments have to be approximated with numerical methods.

A simple, two-parameter PH transform is the *fractional* transform

(A4.9) $$q(x) = \frac{qx}{x+m},$$

with $m \geq 0$ and $0 \leq q \leq 1$. If m is zero, equation (A4.9) equals the constant PH transform. By choosing q and m so that λ remains constant, we can vary the slope of $\lambda(x)$ to a great extent, as shown in the numerical illustration of Section 3.

Other generalized PH transformations[22] will also create steep PRL slopes, but the above fractional transform is sufficient to illustrate the process.

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[22] Since $q(x)$ must be between zero and one, any cumulative or right-hand probability distribution will work, as long as the resulting transformed distribution has a finite mean. Good choices appear to be the right-hand Pareto and exponential distributions. Pursuing these alternatives is worthwhile, but is beyond the scope of this paper.

Appendix 5: Parameter Estimates

Exhibit 1

Calculation of Equity/Liability Ratio and Asset CV
U. S. Property-Liability Industry Data
$Billions

	Calculation of Equity/Liability Ratio	1991	1992	1993	1994	1995	Average	*Source*
A	Loss & LAE Reserve	307.1	326.9	336.3	348.5	360.9		Best's A & A*
B	Acc Year Loss & LAE Incurred	177.3	196.2	175.8	197.4	196.8		Best's A & A
C	Net Liabilities	484.4	523.1	512.1	545.9	557.7		A + B
D	Surplus	158.7	163.1	182.3	193.3	230.0		Best's A & A
E	Surplus/GAAP Equity	0.800	0.800	0.800	0.800	0.800		See Below**
F	GAAP Adjustment	39.7	40.8	45.6	48.3	57.5		D x (1/E - 1)
G	Reserve Discount Factor	0.120	0.120	0.120	0.120	0.120		Exhibit 2
H	Reserve Discount	36.8	39.2	40.3	41.8	43.3		A x G
I	Incurred Loss & LAE Disc Factor	0.079	0.079	0.079	0.079	0.079		Exhibit 2
J	Incurred Loss Discount	14.0	15.5	13.9	15.6	15.6		B x I
K	Economic Liabilities	433.5	468.3	457.8	488.5	498.8		C - H - J
L	Economic Equity	249.3	258.6	282.1	299.1	346.4		D + F + H + J
M	Equity/Liabilities	0.575	0.552	0.616	0.612	0.694	0.610	L/K
	Invested Assets: Portfolio Mix							
N	Stocks	0.183	0.180	0.178	0.184	0.202	0.185	Best's A & A
O	Bonds	0.718	0.710	0.721	0.725	0.701	0.715	Best's A & A
P	Short-Term & Other	0.099	0.110	0.101	0.091	0.097	0.100	Best's A & A
	Total	1.000	1.000	1.000	1.000	1.000	1.000	N + O + P

Invested Assets: Volatility	Q	*Source*
Stocks	0.1776	Exhibit 4
Bonds	0.0545	Exhibit 4
Short-Term & Other	0.0317	Exhibit 4
Total	0.0750	Sum of column Q times average of N, O, P

*Best's Aggregates and Averages

**Source: 1991 testimony of Norman L. Rosenthal in the California Proposition 103 rate hearings (File No. REB 1006; Exhibit 10)

Appendix 5, Exhibit 2

1993 Industry Loss & LAE Reserve Discount Factors
$Millions

Source:
Best's Aggregates and Averages

	A	B	C	D	E	F	G	H	I
	Loss & LAE								
	Paid	Incurred	Unpaid	Ratio	Annual Increment	Unpaid	PV of Unpaid	1 - Ratio	Reserve Discount
						1.0000	0.9208	0.0792	
1993	78,389	188,433	110,044	0.4160	0.4160	0.5840	0.5306	0.0915	10,066
1992	126,143	192,362	66,219	0.6558	0.2398	0.3442	0.3057	0.1120	7,416
1991	131,127	174,809	43,682	0.7501	0.0944	0.2499	0.2207	0.1169	5,106
1990	141,613	170,880	29,267	0.8287	0.0786	0.1713	0.1486	0.1322	3,870
1989	146,034	165,234	19,200	0.8838	0.0551	0.1162	0.0979	0.1572	3,018
1988	132,845	146,379	13,534	0.9075	0.0237	0.0925	0.0773	0.1639	2,219
1987	122,469	132,178	9,709	0.9265	0.0190	0.0735	0.0607	0.1730	1,679
1986	114,811	121,964	7,153	0.9414	0.0148	0.0586	0.0479	0.1838	1,315
1985	112,301	118,318	6,017	0.9491	0.0078	0.0509	0.0417	0.1807	1,087
1984	102,191	107,437	5,246	0.9512	0.0020	0.0488	0.0411	0.1581	830
Prior			24,446		0.0049	0.0439	0.0376	0.1439	3,518
					0.0049	0.0391	0.0340		
					0.0049	0.0342	0.0303		
					0.0049	0.0293	0.0264		
					0.0049	0.0244	0.0224		
					0.0049	0.0195	0.0182		
					0.0049	0.0146	0.0139		
					0.0049	0.0098	0.0094		
					0.0049	0.0049	0.0048		
					0.0049	0.0000			
			K					L	M
Total			334,517		1.0000			0.1199	40,124
			N						
Yield Rate			0.0361						

Column and Item Notes:

- A From Schedule P, Part 1, Column 27
- B From Schedule P, Part 1, Column 12
- C From Schedule P, Part 1, Column 22
- D A / B
- E (D value) - (prior year D value); last ten values are equal, such that the sum is 1
- F 1 - D
- G Present value of remaining column F values using interest rate N; payment is assumed to occur mid-year
- H 1 - G / F
- I C x H
- K Sum of column C
- L M / K
- M Sum of column I
- N Average 3-year Treasury note yield during 1991-95, minus a 2 percentage point risk adjustment

Appendix 5, Exhibit 3

Calculation of CV for Industry Loss & LAE Reserve and Non-Catastrophe Property Losses

Source:
Best's Aggregates and Averages

Year	Industry Loss & LAE Reserve ($Millions)	Growth**	Industry Non-Cat Property* Loss Ratio
1982	111,959		
1983	122,715	0.0961	0.705
1984	134,926	0.0995	0.759
1985	154,425	0.1445	0.719
1986	184,577	0.1953	0.647
1987	217,646	0.1792	0.611
1988	241,692	0.1105	0.634
1989	269,294	0.1142	0.684
1990	289,878	0.0764	0.676
1991	307,141	0.0596	0.631
1992	326,900	0.0643	0.651
1993	336,316	0.0288	0.653
1994	348,504	0.0362	0.689
1995	360,940	0.0357	0.717
1996	365,319	0.0121	0.751
1997	363,351	-0.0054	0.712
Standard Deviation		0.059	0.044

Source: Best's A&A
Cumulative By Line Underwriting Experience -- Industry
Page 156-159; 1993 Edition

	Loss & LAE Ratios				
	Fire	IM	PPAPD	CAPD	Total
1983	62.3	66.5	72.0	73.1	70.5
1984	73.3	70.8	76.4	79.7	75.9
1985	62.0	63.6	75.4	68.4	71.9
1986	54.7	48.5	70.4	55.2	64.7
1987	50.4	46.9	66.8	50.4	61.1
1988	53.9	49.9	69.0	51.9	63.4
1989	66.0	57.0	72.6	56.4	68.4
1990	62.6	59.9	71.1	58.6	67.6
1991	63.1	57.7	65.4	54.6	63.1
1992	76.2	63.1	64.8	57.6	65.1
1993	66.7	62.8	66.7	57.5	65.3
1994	69.9	61.8	70.9	62.1	68.9
1995	69.9	54.1	75.3	67.6	71.7
1996	58.6	62.4	79.6	76.0	75.1
1997	59.6	57.7	74.3	75.9	71.2
NPE					
1983-92	42.5	38.3	255.4	43.3	379.5

*Lines include Fire, Inland Marine, Private Passenger Auto Physical Damage and Commercial Auto Physical Damage

**Equals (current year value) / (prior year value) - 1

Appendix 5, Exhibit 4

Asset CV Estimation

	1946-1995	1 Stocks	2 Bonds	3 90-Day	*Source*
A	Average Return	0.1331	0.0520	0.0484	Exhibit 5
B	Std Deviation	0.1776	0.0993	0.0317	Exhibit 5
C	Std Deviation Adjusted for P-L Portfolio	0.1776	0.0545	0.0317	C1 = B1; C3 = B3; C2 = B2 x H

D	Bond Maturity	20.00	Exhibit 5
E	Bond Duration	14.90	D2 x (1 + A2 x D2/2) / (1 + A2 x D2)
F	Industry Maturity	9.84	From section I below
G	Industry Duration	8.18	F x (1 + A2 x F/2) / (1 + A2 x F)
H	Adjustment Factor	0.549	G / E

I	Maturity Range	1 % of Total	2 Midpoint of Range	
	0 to 1 Year	11.43	0.50	1992 Best's Aggregates and Averages
	1 to 5 Years	24.76	3.00	1992 Best's A & A
	5 to 10 Years	26.27	7.50	1992 Best's A & A
	10 to 20 Years	23.11	15.00	1992 Best's A & A
	Over 20 Years	14.43	25.00	1992 Best's A & A
	Total/Average	100.00	9.84	Sum of (I, column 1) x (I, column 2)

Appendix 5, Exhibit 5

Investment Return Data

Source: Stocks, Bonds, Bills and Inflation: 1996 Yearbook, Ibbotson Associates, Inc. Chicago, 1996

	Annual Total Return*			Index		
	Stocks	Bonds	90-Day Treas	Stocks	Bond	90-Day Treas
1945				3.965	2.930	1.237
1946	-0.081	0.017	0.004	3.645	2.980	1.242
1947	0.057	-0.023	0.005	3.853	2.911	1.248
1948	0.055	0.041	0.008	4.065	3.031	1.258
1949	0.198	0.033	0.011	4.869	3.132	1.272
1950	0.306	0.021	0.012	6.360	3.198	1.287
1951	0.240	-0.027	0.015	7.888	3.112	1.306
1952	0.184	0.035	0.017	9.336	3.221	1.328
1953	-0.010	0.034	0.018	9.244	3.331	1.352
1954	0.526	0.054	0.009	14.108	3.511	1.364
1955	0.316	0.005	0.015	18.561	3.527	1.385
1956	0.066	-0.068	0.025	19.778	3.287	1.419
1957	-0.108	0.087	0.032	17.646	3.573	1.464
1958	0.434	-0.022	0.015	25.298	3.494	1.486
1959	0.120	-0.010	0.030	28.322	3.460	1.530
1960	0.005	0.091	0.027	28.455	3.774	1.571
1961	0.269	0.048	0.021	36.106	3.956	1.604
1962	-0.087	0.079	0.027	32.955	4.270	1.648
1963	0.228	0.022	0.032	40.468	4.364	1.700
1964	0.165	0.048	0.035	47.139	4.572	1.760
1965	0.125	-0.004	0.039	53.008	4.552	1.829
1966	-0.101	0.002	0.048	47.674	4.560	1.916
1967	0.240	-0.049	0.042	59.104	4.335	1.997
1968	0.111	0.026	0.052	65.641	4.446	2.101
1969	-0.085	-0.081	0.066	60.059	4.086	2.239
1970	0.040	0.184	0.065	62.465	4.837	2.385
1971	0.143	0.110	0.044	71.406	5.370	2.490
1972	0.190	0.073	0.038	84.956	5.760	2.585
1973	-0.147	0.011	0.069	72.500	5.825	2.764
1974	-0.265	-0.031	0.080	53.311	5.647	2.986
1975	0.372	0.146	0.058	73.144	6.474	3.159
1976	0.238	0.186	0.051	90.584	7.681	3.319
1977	-0.072	0.017	0.051	84.076	7.813	3.489
1978	0.066	-0.001	0.072	89.592	7.807	3.740
1979	0.184	-0.042	0.104	106.112	7.481	4.128
1980	0.323	-0.026	0.112	140.413	7.285	4.592
1981	-0.048	-0.010	0.147	133.615	7.215	5.267
1982	0.214	0.438	0.105	162.221	10.374	5.822
1983	0.225	0.047	0.088	198.743	10.862	6.335
1984	0.566	0.164	0.099	311.197	12.642	6.959
1985	-0.103	0.309	0.077	279.114	16.549	7.496
1986	0.185	0.198	0.062	330.671	19.833	7.958
1987	0.052	-0.003	0.055	347.967	19.780	8.393
1988	0.168	0.107	0.064	406.458	21.897	8.926
1989	0.315		0.081	534.455		9.651
1990	-0.032		0.075	517.499		10.376
1991	0.305		0.054	675.592		10.938
1992	0.077		0.035	727.412		11.315
1993	0.100		0.030	800.078		11.657
1994	0.013		0.043	810.538		12.157
1995	0.374		0.055	1113.918		12.827

*Calculated by (index, current year)/ (index, prior year) - 1

Stocks are Standard & Poors 500, bonds are 20-year Moody corporate AAA

Appendix 6

Fair Premium and Return on Equity by Layer

Denote the fair premium for the layer from *a* to *b* by $P(a,b)$. The income tax rate is *t*, the risk-free interest rate is *r* and the return on (equity) capital is *R*. Assume that the income tax rate is applied to income measured at the end of the year and that the tax is paid at the end of the year

The fair premium is determined by equation (1.1) with no default present. Thus the fair premium for the layer is

(A6.1) $$P(a,b) = MV(a,b) + T\,C(a,b),$$

where $MV(a,b) = X(a,b)[1+\lambda(a,b)]/(1+r)$ is the market value of the expected loss in the layer, $C(a,b)$ is the amount of capital required for the layer coverage and *T* is the present value of income taxes per unit of capital. Below we show that the second term in equation (A6.1) is the proper cost of income taxes and derive a formula for *T* in equation (A6.4). Substituting this *T* expression into equation (A6.1) and noting that $C(a,b) = c(a,b)\,MV(a,b)$, we get

(A6.2) $$P(a,b) = \frac{X(a,b)[1+\lambda(a,b)]}{1+r} + \frac{X(a,b)[1+\lambda(a,b)]c(a,b)\,r\,t}{(1+r)(1-t)}.$$

Present Value of Income Taxes

We have assumed here a tax system that taxes economic income. We show here that in this case, the only tax burden carried by the insurer is the cost of taxes on investment income from capital.

Assume that the capital is zero and that the premium equals the market value of the loss, denoted by *MVL*. The loss, paid at the end of the year, is $\tilde{L}$. At the end of the year, the income is the premium minus the loss, plus the investment income[23] on the premium. The tax paid is

$$TP = t[MVL(1+r) - \tilde{L}] = tMVL(1+r) - t\tilde{L}. \tag{A6.3}$$

The market value, evaluated at present, of *TP* is the sum of the market values of the two components of equation (A6.3). Since the premium *MVL* is a known quantity, its market value is determined by taking the present value at the risk-free rate. Thus the market value of $tMVL(1+r)$ is $tMVL$. The market value of the random amount of loss $\tilde{L}$ is by definition *MVL*, so the market value of $t\tilde{L}$ is $tMVL$, since t is a constant. Therefore, the market value of *TP* is zero, and the only remaining tax value to consider is due to the income from the capital.

Assume that there is no insurance sold, but the insurer has an amount of capital *C*. The income from the capital is *rC* and the present value of tax on this known amount is $PVC = trC/(1+r)$. Now suppose that somehow the insurer can get the potential policyholders to pay some premium to offset the tax cost of the capital. There is no risk to the transaction, since both the capital and the premium are known. If *PVC* is charged, taxable underwriting income in the amount of *PVC* is generated (there are no losses or expenses). Thus, the additional premium necessary to pay for the taxes on the income from capital is $PVC/(1-t) = trC/[(1+r)(1-t)] = TC$. Therefore,

$$T = \frac{tr}{(1+r)(1-t)}. \tag{A6.4}$$

To check this result, the beginning assets are *C* plus the premium component *TC*. Thus, investment income of $rC(1+T)$ plus underwriting income of *TC* is taxed at the end of the year, leaving an after-tax income of $[1-t][rC(1+T)+TC = rC$. Consequently, the

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[23] We have assumed here that the investment of cash flow is invested in risk-free securities. Myers and Cohn (1987) and Derrig (1994) show that the present value of income taxes on investment income is the same for risky assets as for risk-free assets. This result occurs because the tax paid on risky investment returns is discounted at the risky investment return.

return on capital is r, which is the same return that investors would get in the marketplace for an alternative investment with no risk.

Return on Equity by Layer

At the end of the year, the initial assets (premium plus capital) will have earned expected investment income of $r[P(a,b)+C(a,b)]$ and the expected underwriting income will be $P(a,b)-X(a,b)$. The sum of these two components is taxed at the rate t and then divided by $C(a,b)$ to produce the expected return on capital (equity) $R(a,b)$ for coverage in the layer:

(A6.5) $$R(a,b)=(1-t)\left[r+\frac{P(a,b)(1+r)-X(a,b)}{C(a,b)}\right].$$

Substituting the value of $P(a,b)$ from equation (A6.2) into equation (A6.5) and simplifying, we get

(A6.6) $$R(a,b) \quad = \quad r+(1-t)\frac{1+r}{1+\lambda}\frac{\lambda(a,b)}{c(a,b)}.$$

Using equation (2.6), we get the layer ROE in terms of the layer risk load and the layer beta:

(A6.7) $$R(a,b) \quad = \quad r+(1-t)\left[\frac{1+r}{1+\lambda}\right]\frac{\lambda(a,b)}{c_k+[\beta(a,b)-1]Z_k}.$$

Since the layer risk load and the layer beta can be determined for infinitesimally thin layers, and the other variables in equation (A6.7) are constants, then the layer ROE can also be determined for infinitesimally thin layers. The *point ROE* for the layer at x is

(A6.8) $$R(x) \quad = \quad r+(1-t)\frac{1+r}{1+\lambda}\frac{\lambda(x)}{c_k+[\beta(x)-1]Z_k}.$$

Notice that, since $\lambda(0)=0$, $R(0)=r$, regardless of the layer beta (as long as $c_k \neq Z_k$).